



**IQBAL UNNISA  
W.B. VASANTHA KANDASAMY  
FLORENTIN SMARANDACHE**

**SUPERMODULAR  
LATTICES**

# **Supermodular Lattices**

**Iqbal Unnisa  
W. B. Vasantha Kandasamy  
Florentin Smarandache**

**Educational Publisher Inc.  
Ohio  
2012**

This book can be ordered from:

Education Publisher Inc.  
1313 Chesapeake Ave.  
Columbus, Ohio 43212,  
USA  
Toll Free: 1-866-880-5373

Copyright 2012 by *Educational Publisher Inc.* and the *Authors*

Peer reviewers:

Prof. Mihály Bencze, Department of Mathematics,

Áprily Lajos College, Braşov, Romania

Dr. Sebastian Nicolaescu, 2 Terrace Ave., West Orange, NJ 07052, USA.

Prof. Valeri Kroumov, Okayama Univ. of Science, Okayama, Japan.

Many books can be downloaded from the following

Digital Library of Science:

<http://www.gallup.unm.edu/~smarandache/eBooks-otherformats.htm>

**ISBN-13: 978-1-59973-195-7**

**EAN: 9781599731957**

**Printed in the United States of America**

# CONTENTS

|                                                                |    |
|----------------------------------------------------------------|----|
| Dedication                                                     | 5  |
| Preface                                                        | 7  |
| Chapter One<br><b>PRELIMINARIES</b>                            | 9  |
| Chapter Two<br><b>SIMPLE MODULAR LATTICES OF FINITE LENGTH</b> | 15 |
| Chapter Three<br><b>SUPERMODULAR LATTICES</b>                  | 31 |
| Chapter Four<br><b>SEMI SUPERMODULAR LATTICES</b>              | 51 |

|                                                                    |     |
|--------------------------------------------------------------------|-----|
| Chapter Five                                                       |     |
| <b>SOME INTERESTING EQUATIONAL<br/>CLASSES OF MODULAR LATTICES</b> | 75  |
| Chapter Six                                                        |     |
| <b>SMARANDACHE LATTICES</b>                                        | 85  |
| Chapter Seven                                                      |     |
| <b>GB-ALGEBRAIC STRUCTURES</b>                                     | 99  |
| <b>FURTHER READING</b>                                             | 127 |
| <b>INDEX</b>                                                       | 129 |
| <b>ABOUT THE AUTHORS</b>                                           | 131 |

# DEDICATION



❧ ❧  
*Mr. Mohammed Jalauddin Kokan*  
*and*  
*Mrs. Aaisha Begum*  
❧ ❧

*We dedicate this book to the memory of Dr. Iqbal Unnisa's parents. Her father Mr. Mohammed Jalauddin Kokan was the Post Master in Meenambur and mother Mrs. Aaisha Begum was a housewife. From this remote village of Meenambur in Tamilnadu, her parents shifted to Chennai to educate their children.*

*In those days, girl children being enrolled in primary school was rare. Yet, the Kokans succeeded in educating their daughters into doctors and doctorates. In fact, Dr. Iqbal Unnisa has the distinction of being the first Muslim women in India to receive her doctoral degree in Mathematics.*

*This dedication is a gesture of honouring Mr. Mohammed Jalauddin Kokan and Mrs. Aisha Begum, and at the same time, acknowledging the great role that education plays in the empowerment of women.*

## PREFACE

In lattice theory the two well known equational class of lattices are the distributive lattices and the modular lattices. All distributive lattices are modular however a modular lattice in general is not distributive.

In this book, new classes of lattices called supermodular lattices and semi-supermodular lattices are introduced and characterized as follows:

A subdirectly irreducible supermodular lattice is isomorphic to the two element chain lattice  $C_2$  or the five element modular lattice  $M_3$ .

A lattice  $L$  is supermodular if and only if  $L$  is a subdirect union of a two element chain  $C_2$  and the five element modular lattice  $M_3$ .

A modular lattice  $L$  is  $n$ -semi-supermodular if and only if there does not exist a set of  $(n + 1)$  elements  $a_1, a_2, \dots, a_n$  in  $L$  such that  $a + a_1 = a + a_2 = \dots = a + a_n > a$  with  $a > a_i a_j$  ( $i$  not equal  $j$ );  $i, j = 1, 2, \dots, n$ .

A modular lattice  $L$  is  $n$ -semi-supermodular if and only if it does not contain a sublattice whose homomorphic images is isomorphic to  $M_{i_1+2, i_2+2, \dots, i_r+2}$  or  $\widehat{M}_{i_1+2, i_2+2, \dots, i_r+2}$  with  $i_1 + i_2 + \dots + i_r = n - 1$ ;  $i, j \geq 1$ .



We define the concept of Smarandache lattices and GB-algebraic structures are characterized in chapters six and seven respectively.

This book has seven chapters. Chapter one is introductory in nature. A simple modular lattice of finite length is introduced and characterized in chapter three. In chapter four the notion of supermodular lattices is introduced and characterized and chapter five introduces the notion of  $n$ -semi-supermodular lattices and characterizes them.

It is pertinent to keep on record part of this book is the second authors Ph.D thesis done under the able guidance of the first author Late Professor Iqbalunnisa. Infact the last two authors where planning for a lattice theory book a year back but due to other constraints we could not achieve it. Now we with a heavy heart have made this possible.

We thank Dr. K.Kandasamy for proof reading and being extremely supportive.

IQBAL UNNISA  
W.B.VASANTHA KANDASAMY  
FLORENTIN SMARANDACHE

## Chapter One

# PRELIMINARIES

In this chapter we recall some definitions and results which are made use of throughout this book. The symbols  $\leq$ ,  $\geq$ ,  $+$ ,  $\cdot$  will denote inclusion, noninclusion, sum (least upper bound) and product (greater lower bound) in any lattice  $L$ ; while the symbols,  $\subseteq$ ,  $\cup$ ,  $\cap$ ,  $\in$ ,  $\notin$  will refer to set inclusion, union (set sum), intersection (set product), membership, and non-membership respectively. Small letters  $a$ ,  $b$ , ... will denote elements of the lattice and greek letters,  $\theta$ ,  $\phi$ , ... will stand for congruences on the lattice.

A binary relation  $\theta$  on  $L$  is said to be an equivalence relation if it satisfies.

- (i)  $x \equiv x \ (\theta)$  (reflexive)
- (ii)  $x \equiv y \ (\theta) \Rightarrow y \equiv x \ (\theta)$  symmetric
- (iii)  $x \equiv y \ (\theta); y \equiv x \ (\theta) \Rightarrow x \equiv z \ (\theta)$  (transitive)

If it further satisfies the substitution property.

- (iv)  $x \equiv x' \ (\theta); y \equiv y' \ (\theta) \Rightarrow x + y \equiv x' + y' \ (\theta)$  then it is called an additive congruence. An equivalence relation which has the substitution property.
- (v)  $x \equiv x' \ (\theta); y \equiv y' \ (\theta) \Rightarrow xy \equiv x'y' \ (\theta)$  is called a multiplicative congruence. If a binary relation satisfies

the conditions (1) to (v) then it is said to be a lattice congruence or merely a congruence on  $L$ .

**Result 1.1:** If  $a = b$  ( $\theta$ ) in a lattice  $L$  then  $x = y$  ( $\theta$ ) for all  $x, y$  in  $L$  such that  $ab \leq x, y \leq a + b$  [2].

Let  $a, b$  in  $L$  such that  $a \geq b$ . Then the set of all elements  $x$  in  $L$  such that  $a \geq x \geq b$  is called the interval  $(a, b)$ . If  $a = b$ ,  $(a, b)$  is called a prime interval. If  $a = b$ ,  $(a, b)$  is called a trivial interval.

If  $a \equiv b$  ( $\theta$ ) ( $a, b$  in  $L$ ;  $a \geq b$ ) then  $x \equiv y$  ( $\theta$ ) for all  $x, y$  in  $L$  such that  $a \geq x, y \geq b$  (by result 1.1) and  $\theta$  is said to annul the interval  $(a, b)$ .

Intervals of the form  $(x, x+y)$  and  $(xy, y)$  are said to be perspective intervals. We say

$$(xy, y) \nearrow (x, x+y) \text{ or equivalently } (x, x+y) \searrow (xy, y).$$

If  $I = (a, b)$  is an interval of  $L$  then the interval  $(a+x, b+x)$  for any  $x$  in  $L$  is called an additive translate of the interval  $I$  and is written as  $I+x$ ; and the interval  $(ax, bx)$  for any  $x$  in  $L$  is called a multiplicative translate of the interval  $I$  and is written as  $Ix$ .

An interval  $J$  is the lattice translate of an interval  $I$  of  $L$  if elements  $x_1, x_2, \dots, x_n$  can be found such that

$$J = (((I + x_1)x_2) + x_3) \dots x_n \text{ or}$$

$$J = (((I.x_1) + x_2) x_3) \dots x_n,$$

where  $n$  is finite and  $+$ ,  $.$  occur alternatively.

**Result 1.2:** Lattice translation is a transitive relation.

**Result 1.3:** Any lattice translate of an interval  $I$  of a distributive lattice  $L$  can be written as  $(I + y)x$  or  $(I.x) + y$  for some  $x, y$  in  $L$  [10].

**Result 1.4:** Any subinterval  $J$  of an interval  $I$  is a lattice translate of  $I$  [10].

**Result 1.5:** Any congruence  $\theta$  on  $L$  which annuls  $I$  annuls all lattice translates of  $I$  [10].

**Result 1.6:** Any non-null set  $S$  in  $L$  is a congruence class under some congruence relation on  $L$  if and only if

- (i)  $S$  is a convex sublattice and
- (ii) Lattice translate of intervals in  $S$  lie wholly within  $S$  or outside  $S$  [10].

**DEFINITION 1.1:** *The smallest congruence which annuls a set  $S$  of a lattice  $L$  is called the congruence generated by the set  $S$ .*

**Result 1.7:** Let  $I$  be an interval of a lattice  $L$ .  $\theta_I$  the congruence generated by  $I$  in  $L$ .  $x \equiv y (\theta_I)$  if and only if there exists a finite set of elements  $x+y = x_0 > x_1 > x_2 \dots > x_n = xy$  such that  $(x_i, x_{i+1})$  is a lattice translate of the interval  $I$  [10].

**Result 1.8:** The lattice translate of a prime interval in a modular lattice can only be a prime interval [10].

**Result 1.9:** Let  $L$  be a modular lattice.  $I$  and  $J$  intervals of  $L$  such that  $I$  is a lattice translate of  $J$  then  $I$  is projective with a subinterval of  $J$  [10].

**DEFINITION 1.2:** *The modular lattice consisting of  $n+2$  elements ( $n \geq 3$ ),  $a, x_1, \dots, x_n, b$  satisfying  $x_i + x_j = a$  ( $i \neq j, i, j = 1, 2, \dots, n$ ),  $x_i x_j = b$  for all  $i, j = 1, 2, \dots, n; i \neq n$  is denoted by  $M_n$ . The modular lattice consisting of the elements  $a, x_1, x_2, \dots, x_n, b, y_1, \dots, y_{m-1}, c$  satisfying*

$$x_i + x_j = a \text{ for all } i \neq j, i, j = 1, 2, \dots, n$$

$x_i x_j = b$  for all  $i, n \neq j, i, j = 1, 2, \dots, n$   
 $b + y_i = x_n$  for all  $i = 1, 2, \dots, m-1$   
 $y_i + y_j = x_n$  for all  $i \neq j, i, j = 1, 2, \dots, m-1$   
 $b y_i = c$  for all  $i = 1, 2, \dots, m-1$   
 $y_i y_j = c$  for all  $i, j = 1, 2, \dots, m-1$  is denoted by  $M_{n,m}$ .

We denote  $M_{n_1, n_2, \dots, n_k}$  ( $n_k \geq 3$ ) in a similar fashion.

We define  $\hat{M}_{m,n,r}$  to be the modular lattice consisting of the elements  $a, x_1, \dots, x_n, b, y_1, \dots, y_{m-1}, c, z_1, z_2, \dots, z_{r-1}, d, cd$  such that  $a, x_1, \dots, x_n, b$  form a lattice isomorphic to  $M_n$ ;  $x_1, b, y_1, \dots, y_{m-1}, c$  form a lattice isomorphic to  $M_m$ ;  $x_2, b, z_1, \dots, z_{r-1}, d$  form a lattice isomorphic to  $M_r$ . We extend definition to  $\hat{M}_{n,m,r,s}$  in two ways.

First  $\hat{M}_{n,m,r,s}$  is got by taking elements  $a, x_1, x_2, \dots, b, y_1, \dots, y_{m-1}, b_1, z_1, \dots, z_{r-1}, b_2, u_1, \dots, u_{s-1}, b_3, b_1 b_2, b_1 b_3, b_3 b_1, b_1 b_2 b_3$  such that  $a, x_1, x_2, \dots, x_n, b$  form a lattice isomorphic to  $M_n$ ;  $x_1, b, y_1, \dots, y_{m-1}, b_1$  form a lattice isomorphic to  $M_m$ ;  $x_2, b, z_1, \dots, z_{r-1}, b_2$  form a lattice isomorphic to  $M_r$ ;  $x_3, b, u_1, \dots, u_{s-1}, b_3$  form a lattice isomorphic to  $M_s$ .

In the second; we define  $\hat{M}_{m,n,r,s}$  by taking the elements

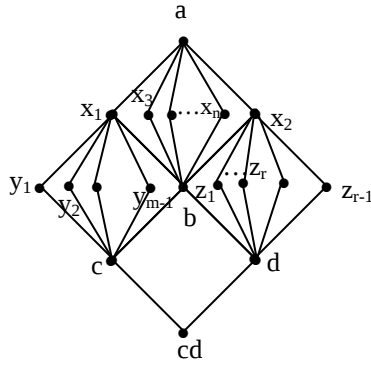
$a, x_1, \dots, x_n, b, y_1, \dots, y_{m-1}, b_1, z_1, z_2, \dots, z_{r-1}, b_2, u_1, \dots, u_{s-1}, b_3, b_1 b_2, b_2 b_3$  such that  $a, x_1, \dots, x_n, b$  form a sublattice isomorphic to  $M_n$ ,

$x_1, b, y_1, \dots, y_{m-1}, b_1$  form a sublattice isomorphic to  $M_m$ ,

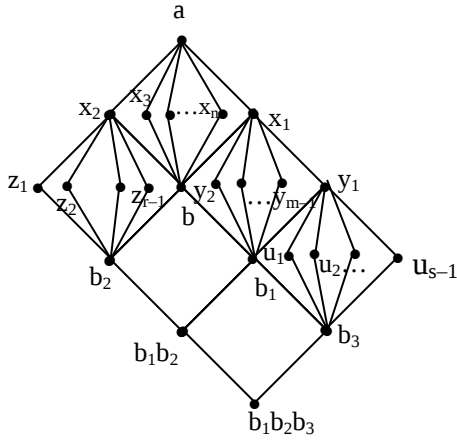
$x_2, b, z_1, z_2, \dots, z_{r-1}, b_2$  form a sublattice isomorphic to  $M_r$ ,

$y_1, b_1, u_1, \dots, u_{s-1}, b_3$  form a sublattice isomorphic to  $M_s$ .

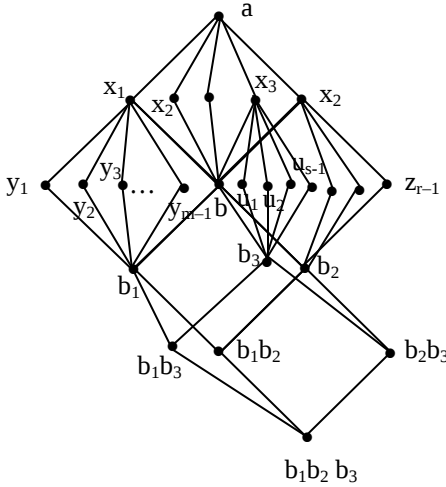
$\check{M}_{m,n,r,s} \dots$  is defined dually.



$\hat{M}_{m,n,r}$  **Figure 1.1**



$\hat{M}_{n,m,r,s}$  **Figure 1.2**



**Figure 1.3**

These are illustrated in figures 1.1, 1.2 and 1.3.

**DEFINITION 1.3:** Let  $\theta$  be a congruence on a lattice  $L$ .  $\theta$  is called separable if and only if for every pair of comparable elements  $a < b$  there exist a finite sequence of elements.

$$a = x_0 < x_1 < \dots < x_n = b$$

such that either  $(x_{i-1}, x_i)$  is annulled by  $\theta$  or  $(x_{i-1}, x_i)$  consist of single point congruence classes under  $\theta$ .

**THEOREM [4]:** Let  $L$  be a modular lattice and let  $C_0$  and  $C_1$  be chains in  $L$ . The sublattice of  $L$  generated by  $C_0$  and  $C_1$  is distributive.

**THEOREM [6, 7, 13]:** Any subdirectly irreducible modular lattice of length  $n \geq 3$  has a sublattice whose homomorphic image is isomorphic to  $M_{3,3}$ .

## Chapter Two

# SIMPLE MODULAR LATTICES OF FINITE LENGTH

In the first chapter we defined the notions of additive translate, multiplicative translate and lattice translate of an interval. We introduce in this chapter, the notion of a distributive translate of an interval and study the properties of distributive translates of prime intervals in a modular lattice. This study leads us to a characterization of simple modular lattices of finite length.

Throughout this chapter  $L$  will denote a modular lattice, unless otherwise stated.

**DEFINITION 2.1:** A prime interval  $I$  of  $L$  is said to be distributive if for all nontrivial intervals  $J = (I+x)y$  there exist  $p, q \in L$  with  $J = Ip + q$  for all the nontrivial intervals  $J_1 = Ix_1 + y_1$  there exist  $p_1, q_1$  in  $L$  with  $J_1 = (1 + p_1)q_1$ .

**Lemma 2.1:** Any prime interval of a distributive lattice is distributive.



**Proof:** Follows as  $J = (I+x)y$  implies  $J = Iy + xy$  and  $J_1 = Ix_1 + y_1$  implies  $J_1 = (I+y_1)(x_1 + y_1)$ .

**Lemma 2.2:** In a distributive lattice if a nontrivial interval  $J = (I+x)y$ , then  $Iy$  is nontrivial.

**Proof:** Now  $J = Iy + xy$  and if  $Iy$  were trivial, so would be  $J$ .

**DEFINITION 2.2:** If  $I = (a, b)$  and  $J = (c, d)$  then the intervals  $(a + c, b + d)$  and  $(ac, bd)$  are denoted by  $I + J$  and  $IJ$  respectively.

**Lemma 2.3:** If a nontrivial interval  $J = (I + x)y$  then  $I + J$  cannot be trivial.

**Proof:** Let  $I = (a, b)$  and  $J = (c, d)$  then  
 $c = (a + x)y$ ;  $d = (b + x)y$ .

Also  $a+c = a + (a+x)y = (a+x)(a+y)$  and  
 $b+d = b+(b+x)y = (b+x)(b+y)$ .

Now if  $I + J$  is trivial then  $a + c = b + d$  that is  
 $(a+x)(a+y) = (b+x)(b+y)$  which means

$$\begin{aligned} a + x &= (a + x) + (a + x)(a + y) \\ &= (a + x) + (b + x)(b + y) \\ &= (b + x)(a + x + b + y) \\ &= b + x. \end{aligned}$$

So  $(a + x)y = (b+x)y$  that is  $J$  is trivial; a contradiction.

Dually we have

**Lemma 2.4:** If any nontrivial  $J = Ix + y$  then  $IJ$  cannot be trivial.

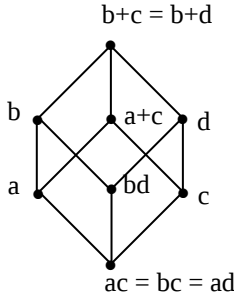
**Lemma 2.5:** If  $I = (a, b)$  is a prime interval and  $J = (c, d) = (I + x)y$  then  $I$  is distributive if and only if  $Iy$  is nontrivial and  $J = Iy + c$ . (Equivalently  $J = Id + c$ ).

**Proof:** Under the hypothesis of the lemma let  $I = (a, b)$  be distributive then  $J = I_p + q$  for some  $p, q$  in  $L$ , then by Lemma 2.4  $IJ$  cannot be trivial.

$$\begin{aligned} \text{Now } IJ &= (a \ (a+x)y, b(b+x)y) \\ &= (ay, by) \\ &= I_y. \end{aligned}$$

Hence  $I_y$  is nontrivial.

Conversely let  $I_y$  be nontrivial, then as the lattice  $L$  is modular and  $I_y$  is a lattice translate of a prime interval, it can only be a prime interval as shown in figure 2.1.



**Figure 2.1**

$$\begin{aligned} I_y &= (ay, by) \\ &= (a(a+x)y, b(b+x)y) \\ &= (ac, ad). \end{aligned}$$

Hence  $ac$  is covered by  $bd$  ( $I_y$  is a prime interval). Now  $L$  is a modular lattice of finite length hence we can define a dimension function  $d$  which satisfies  $d(a+b) + d(ab) = d(a) + d(b)$ . Thus we have

$$\begin{aligned} d(a+c) + d(ac) &= d(a) + d(c) \\ d(b+d) + d(bd) &= d(b) + d(d). \end{aligned}$$

Subtracting the former from the latter we get

$$\begin{aligned} [d(b+d) - d(a+c) + [d(bd) - d(ac)] \\ = [d(b) - d(a)] + [d(d) - d(c)] \end{aligned}$$

$$= 1 + 1 \text{ (as } b \text{ covers } a \text{ and } d \text{ covers } c).$$

$$= 2.$$

Now  $d(bd) - d(ac) = 1$ , implies  $d(b+d) - d(a+c) = 1$  that is  $b + d$  covers  $a + c$ . Now  $a + c \leq b + c \leq c \leq b + d$  and  $a + c$  is covered by  $b + d$ .

Either  $b + c = a + c$  or  $b + c = b + d$ .

If  $a + c = b + c$  then  $a + c > b$

$$\begin{aligned} \Rightarrow a + (a + x)y &> b \\ \Rightarrow (a + x)(a + y) &> b \\ \Rightarrow (a + x) &> (a + x)(a + y) > b \\ \Rightarrow a + x &> b + x \\ \Rightarrow a + x &= b + x \\ \Rightarrow (a + x)y &= (b + x)y \\ \Rightarrow J &\text{ is trivial. A contradiction. Thus } b + c = b + d. \end{aligned}$$

Now  $ac \leq bc \leq bd$  and  $ac$  is covered by  $bd$ . So  $bc = ac$  or  $bc = bd$ .  $bc \neq bd$ . For if  $b + c = b + d$ , as  $c < d$  and the lattice is modular, we have  $bc = ac$ .

Next  $ad = a(b + x)y = ay = ac$ . That is  $ad = ac$ .

Now,

$$\begin{aligned} abd &= bad = bac = bbc = bc = ac \\ ac &= ad = abd = bc \\ \text{that is } abd &= bcd. \end{aligned}$$

So  $ac = bcd$  which can be written as  $a c c = b d c$ . Now  $ac \leq bd$ . So  $ac + c \neq bd + c$  ( $L$  is modular) that is  $c \neq c + bd$ . So  $c \leq bd + c \leq d$ . Also  $c$  is covered by  $d$  so  $bd + c = d$  that is  $J = Id + c = Iy + c$ .

Dually we have

**Lemma 2.6:** If  $I = (a, c)$  is a prime interval and  $J = (c, d) = Ix + y$  then  $J$  is a distributive lattice translate of  $I$  if and only if  $I + y$  is nontrivial and  $J = (I + y)d$ . Equivalently  $J = (I + c)d$ .

**Lemma 2.7:** If  $a$  is covered by  $b$  and  $c$  is covered by  $d$  with  $b \leq c$  then  $(c, d) \neq ((a+x)y, (b+x)y)$  for any  $x, y \in L$ .

**Proof:** If  $(c, d) = ((a+x)y, (b+x)y)$  and  $a$  is covered by  $b$  and  $b \leq c$  and  $c$  is covered by  $d$  then  $c = (a+x)y$ .

$$\begin{aligned} c+x &= (a+x)y+x \\ &= (a+x)(x+y) \\ (c+x)y &= (a+x)y = c. \end{aligned}$$

But  $(b+x)y$  lies between  $(a+x)y$  and  $(c+x)y$  as  $a < b \leq c$ , so equals  $c$ , a contradiction,  $a \neq d$  and  $(b+x)y = d$ .

**Lemma 2.8:** Let  $a$  is covered by  $b$  and  $c$  is covered by  $d$  with  $d \leq a$  then  $(c, d) \neq ((a+x)y, (b+x)y)$  for any  $x, y$  in  $L$ .

**Proof:** If  $d = (b+x)y$  then  $d+x = (b+x)y+x$   
 $= (b+x)(x+y)$  implies  
 $(d+x)y = (b+x)y = d$ .  
 Thus  $d = (d+x)y < (a+x)y < (b+x)y = d$  implies  
 $(a+x)y = d = c$ ; a contradiction.

**Corollary 2.1:** If there exists intervals  $I = (a, b)$ ,  $J = (c, d)$  with  $a < b \leq c < d$  then neither can be a distributive translate of the other in any general lattice.

**Corollary 2.2:** If  $a < b \leq c < d$  in a distributive lattice then neither  $J = (a, b)$  nor  $J = (c, d)$  can be a lattice translate of the other.

**DEFINITION 2.3:** A prime interval  $J = (c, d)$  of a modular lattice  $L$  is called a distributive lattice translate of a prime interval  $I = (a, b)$  if and only if  $J$  can be expressed as  $(I + c)d$  or equivalently  $J$  can be expressed as  $Id + c$ .

**Lemma 2.9:** If  $(I + x)y = J$  with  $IJ = Iy$  trivial then  $I + J$  contains a five element modular sublattice and is of length 2.

**Proof:** Length  $(I + J) + \text{Length}(IJ) = 2$ . As Length  $IJ = 0$  (given); Length  $(I + J) = 2$ .

Let  $I = (a, b)$  and  $J = (c, d)$ ,  $IJ = 0$  implies  $ac = bd$ . Also  $ac \leq bc \leq bd$  and  $ac \leq ad \leq bd$ . So  $ac = bc$  and  $ad = bd$ . Now  $L$  is modular. Therefore  $a + c \neq b + c$  and  $a + d \neq b + d$ . Also  $ac = ad$  and  $bc = bd$ . Again as  $L$  is modular  $a + c \neq a + d$  and  $b + c \neq b + d$ .

Next  $b + c \neq a + d$ . For if  $b + c = a + d$  then  $b + c + a + d = a + d$  then is  $b + d = a + d$ ; a contradiction.

But  $a + c$  is covered by  $b + c$  and  $b + c$  is covered by  $b + d$  and  $a + c$  is covered by  $a + d$  and  $a + d$  is covered by  $b + d$ .

Also  $a + c = (a + x)(a + y)$  and  $b + c = (b + x)(b + y)$ . Let  $p = (a + x)(b + y)$  as  $b + d = (b + x)(b + y)$  either  $(p, b + d)$  is trivial or  $p$  is covered by  $b + d$ . If  $p = b + d$ , then  $(a + x)(b + y) = (b + x)(b + y)$ . So  $(a + x)(b + y)y = (b + x)(b + y)y$  will imply  $(a + x)y = (b + x)y$  that is  $c = d$ ; a contradiction.

So  $(p, b + d)$  is nontrivial. Thus  $p$  is covered by  $b + d$ . Now  $b + d$  covers  $b + c$  and  $a + d$ . We assert neither  $b + c = p$  nor  $a + d = p$ . For if  $b + c = p$  then  $(b + c, b + d) = (I + x)(b + d)$  and  $a$  is covered by  $b \leq b + c$  and  $b + c$  is covered by  $b + d$ ; a contradiction in view of Lemma 2.7.

If  $a + d = p$  then  $(a + d, b + d) = [J + a + x](b + d)$  and  $c$  is covered by  $d \leq a + d$  and  $a + d$  is covered by  $b + d$ ; a contradiction in view of Lemma 2.8.

Thus  $a + c, b + c, p, a + d, b + d$  is isomorphic to the five element modular lattice of length 2 contained in  $I + J$  (cf. figure 2.2)

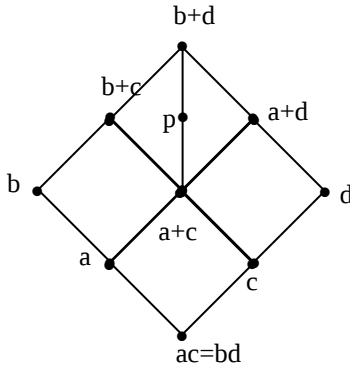


Figure 2.2

Dually we have

**Lemma 2.10:** If  $Ix + y = J$  with  $I + J = I + y$  trivial then  $IJ$  contains a five element modular sublattice and is of length 2.

**Lemma 2.11:** Let  $L$  be a modular lattice of finite length. Let  $C: 0 = x_0 < x_1 < x_2 < \dots < x_n = 1$  be a maximal chain connecting 0 and 1. Let  $I = (a, b)$  ( $a$  is covered by  $b$ ) be an arbitrary prime interval of  $L$  then  $I$  is a distributive lattice translate of a unique prime interval of  $C$ .

**Proof:** Now  $I = ((0, 1) + a) b$  as  $(x_0 + a)b = a$  and  $(x_n + a)b = b$ . Next  $a \leq (x_i + a) b \leq b$  for all  $x_i$ . Given  $a$  is covered by  $b$ ; either  $(x_1 + a) b = a$  or  $b$ ; for all  $x_i$ .

Let  $k$  be the largest  $i$  for which  $(x_k + a)b = a$ ; then  $(x_{k+1} + a)b = b$ .

So  $I = ((x_k, x_{k+1}) + a)b$ .

Now as  $L$  is modular and  $a$  is covered by  $b$  we have  $I = ((x_k, x_{k+1}) b) + a$  also.

Thus  $I = (a, b)$  is a distributive lattice translate of  $(x_k, x_{k+1})$ .

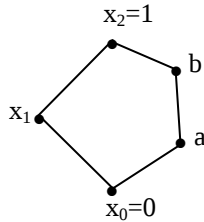
Further this  $(x_k, x_{k+1})$  is unique. For if  $I$  is a distributive lattice translate of another  $(x_j, x_{j+1})$  of  $C$  then

$$I = (a, b) = ((x_j, x_{j+1}) + a) b \text{ by Lemmas 2.5 and 2.6.}$$

Now as  $k$  and  $j$  are comparable without loss in generality we can assume  $k > j$ , then we see that  $(x_j + a) b = (x_{j+1} + a) b = a$ ; a contradiction. Thus  $k = j$  and so the uniqueness of the interval is established.

**Remark:** The modularity of the lattice is a necessary condition in Lemma 2.11.

**Proof:** Consider the lattice of figure 2.3



**Figure 2.3**

and the chain  $C : 0 = x_0 < x_1 < x_2 = 1$ . Let  $I = (a, b)$  be the prime interval. Now  $I$  is not a distributive lattice translate of any interval in  $C$ .

$$\begin{aligned} \text{For } ((x_0, x_1) + a)b &= (a, b) \neq (x_0, x_1)b + a \\ ((x_1, x_2)b) + a &= (a, b) \neq [(x_1, x_2) + a]b. \end{aligned}$$

Let  $L$  be a modular lattice of length  $n$  and  $C : 0 = x_0 < x_1 < \dots < x_n = 1$  be a maximal chain connecting 0 to 1. Let  $P$  denote the totality of all prime intervals of  $L$ . Partition  $P$  with respect to  $C$  into  $n$  classes  $P_1, P_2, \dots, P_n$  by the following procedure.

$P_i$  = the set of all intervals  $P$  which are distributive lattice translates of  $(x_{i-1}, x_i)$ .

Now it follows as a consequence of Lemma 2.11 above that any prime interval of  $L$  belongs to one and only one class  $P_i$ .

**Lemma 2.12:** If  $L$  is a distributive lattice then for each interval  $J_i = (x_{i-1}, x_i)$  of  $C$ ,  $\theta_{J_i}$  annuls just those prime intervals of  $L$  which belong to  $P_i$  and no more.

**Proof:** Follows as any lattice translate of any prime interval  $J_i$  of  $L$  is a distributive lattice translate of  $J_i$ .

**Lemma 2.13:** If  $J_i$  is a distributive interval of a modular lattice  $L$  then  $\theta_{J_i}$  annuls just those prime intervals of  $L$  which belong to  $P_i$  and no more.

**Proof:** Follows as in the case of Lemma 2.12.

**Lemma 2.14:** If a prime interval  $I = (a, b)$  of  $L$  is a distributive lattice translate of some interval  $J_i = (x_{i-1}, x_i)$  of  $C$  then  $I$  is a lattice translate of  $J_i$  considered as intervals in the distributive sublattice generated by  $I$  and  $C$ .

**Proof:** As in this case

$$\begin{aligned} I &= ((x_{i-1}, x_i) + a)b \\ &= ((x_{i-1}, x_i)b) + a \end{aligned}$$

and all the elements involved belong to the sublattice generated by  $I$  and  $C$ .

**Lemma 2.15:** If  $I = (a, b)$ ,  $I_1 = (a_1, b_1)$  are prime intervals of  $L$  such that  $a$  is covered by  $b$  and  $b \leq a_1$  is covered by  $b_1$  then the classes to which  $I$  and  $I_1$  belong in the partition of  $P$  with respect to any arbitrary chain  $C$  are distinct.



**Proof:** Let  $D$  denote the chain of  $L$  consisting of the intervals  $I$  and  $I_1$ ; and let  $S$  be the distributive sublattice generated by  $C$  and  $D$ . Let  $P_1, P_2, \dots, P_n$  be the classes corresponding to the partitioning of the prime intervals of  $L$  with respect to the chain  $C$ ; and  $P'_1, P'_2, \dots, P'_n$  be the classes corresponding to the partitioning of the prime intervals of  $S$  with respect to the chain  $C$ .

In view of Lemma 2.14 observe that the interval  $I$  belongs to  $P_1$  (i.e., the class containing  $(x_{i-1}, x_i)$ ) if and only if  $I$  belongs to  $P'_1$  (i.e., the class  $(x_{i-1}, x_i)$ ).

Now as  $S$  is a distributive lattice,  $I$  and  $I_1$  cannot be lattice translates of each other in  $S$ ; hence will belong to different classes under the partitioning of  $S$  with respect to  $C$  which in turn gives the required result.

**Lemma 2.16:** If  $C_1$  is any other maximal chain connecting 0 to 1 of  $L$  then  $C_1$  has exactly  $n$  prime intervals each of them belong to the classes  $P_1, P_2, \dots, P_n$ , taken in some order.

**Proof:** Follows from Lemma 2.15.

**Lemma 2.17:** If a lattice translate  $K = (J_i + x)y$  or  $(J_i x + y)$  of  $J_i$  is non-distributive then  $K$  belongs to a class  $P_j$  different from  $P_i$ . Further the prime intervals of the five element sublattice  $K + J_i$  ( $KJ_i$ ) belong either  $P_i$  or to  $P_j$ .

**Proof:** Lemma 2.9 and Lemma 2.10 assert the existence of the five element sublattice  $K + J_i$  (and  $KJ_i$ ) respectively consider any maximal chain  $C_1$  of  $L$  connecting 0 and 1 which passes through the end points of  $K + J_i$  (or  $KJ_i$ ). From the previous lemma it follows that the prime intervals of  $C$  except those within  $K+J_i$  (or  $KJ_i$ ) belong to  $(n-2)$  classes of the partitioning of  $L$  with respect to  $C$ . Let the two classes which are omitted be  $P_i$  and  $P_j$ . These are the classes to which the prime intervals in  $K + J_i$  (or  $KJ_i$ ) belong irrespective of the element of  $K + J_i$  (or  $KJ_i$ ) occurring in the chain  $C_1$ .

**Lemma 2.18:** If the lattice translates of  $(x_{i-1}, x_i)$  and  $(x_{i-1}, x_j)$  of the previous lemma are contained completely within the classes  $P_i$  and  $P_j$  then  $\theta_{(x_{i-1}, x_i)}$  annuls just the prime intervals belonging to these two classes.

**Proof:** Follows as  $\theta_{(x_{i-1}, x_i)}$  being the smallest congruence annulling  $(x_{i-1}, x_i)$  annuls just those intervals which can be written as a finite sum of lattice translates of  $(x_{i-1}, x_i)$ .

Let  $K_1$  be a non-distributive lattice translate of  $(x_{i-1}, x_i)$  or  $(x_{j-1}, x_j)$  lying outside the classes  $P_i$  and  $P_j$ , then the classes  $P_i$  and  $P_u$  (or  $P_j$  and  $P_u$ ) meet in a five element modular lattice (where  $P_u$  denotes the class to which  $K_1$  belongs). Thus to pass from one of the classes  $P_i$  to another class  $P_k$  one has to pass through a five element modular lattice.

Hence we have

**THEOREM 2.1:** A modular lattice  $L$  of finite length  $n$  is simple if and only if the partition of  $L$  with respect to some arbitrary chain  $C$  satisfies property  $(\alpha)$ .

“Any two of the classes  $P_1, P_2, \dots, P_n$  in the partitioning can be linked to one another by a sequence such that any two consecutive classes of the sequence meet at a five element modular lattice”.

**Proof:** Let  $C : 0 = y_0 < y_1 < \dots < y_n = 1$  be the chain. If  $L$  is simple then every prime interval of  $L$  is a lattice translate of any other prime interval. Now the prime interval  $J = (y_1, y_2)$  of  $L$  is a lattice translate of  $I = (y_0, y_1)$ . Let

$$J = (I + x_1)x_2 + x_3 \dots + x_{2r}$$

be a representation of  $J$  as a lattice translate of  $I$ . Let this representation be one of those which cannot be further reduced; that is  $(I + x_1)x_2$  is a nondistributive translate of  $I$ ,  $(I + x_1)x_2 + x_3$  is a nondistributive translate of

$$I + x_1 \dots ((I + x_1)x_2 + \dots + x_{2i-2}) + x_{2i-1}$$

is a nondistributive translate of

$$(I + x_1) x_2 + x_3 \dots) + x_{2i-3} \dots \text{etc.}$$

Let  $I, I+x_1 \in P_{\alpha_1} = P_1$

$$((1+x_1) x_2, (1+x_1) x_2 + x_3 \in P_{\alpha_2}$$

$$((1+x_1) x_2+x_3)x_4, ((I+x_1)x_2 + x_3)x_4 + x_5 \in P_{\alpha_3}$$

...

$$J \in P_{\alpha_r} = P_2.$$

$I \in P_1 = P_{\alpha_1}, P_{\alpha_2}, \dots, P_{\alpha_r} = P_2$ ;  $J \in P_2$  is the sequence by which these are linked. Similarly any  $P_i$  and  $P_j$  would be linked.

Conversely if a modular lattice  $L$  satisfies property  $(\alpha)$  then any two of the prime intervals of  $L$  are lattice translates of each other and hence  $L$  is simple.

**Remark:** Start with any class  $P_1$ . Now this class  $P_1$  should be linked to some class  $P_{\alpha_2}$  so we should pass through a five element modular lattice. If those two are not further linked we would set a congruence on  $L$  just annulling these two classes and hence  $L$  will not be simple. So in the class of a simple lattice, atleast one of  $P_1$  or  $P_{\alpha_2}$  should be linked to another class  $P_{\alpha_3}$  and this will give another five element modular lattice etc. This process will continue until all the classes are exhausted and so, we would atleast have  $(n-1)$  such five element modular lattices existing in  $L$ .

Conversely if we have a modular lattice  $L$  containing  $(n-1)$  such five element modular lattice in such a way that these link any two of the classes  $P_i$  with respect to some chain  $C$  then  $L$  is simple.

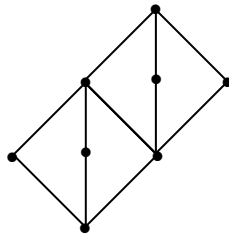
**Corollary 2.3:**  $L$  is a modular lattice containing a maximal chain  $C$  such that any two of the classes of the partition of  $P$  with respect to  $C$  satisfy property  $(\alpha)$  then the classes of the partition of  $P$  with respect to any other chain  $C_1$  also satisfy property  $(\alpha)$ .

**Proof:** This follows as the first condition implies the simplicity of the lattice and the second is obtained as the choice of the chain  $C$  in the previous theorem is arbitrary.

**Corollary 2.4:** If  $L$  is a simple modular lattice then every class  $P_i$  has atleast one direct link with some other class  $P_j$ .

**Corollary 2.5:** Let  $L$  is a simple modular lattice of length  $n$  with  $n \geq 3$ , then there exists atleast one class  $P_i$  which has direct links with two or more classes.

**THEOREM 2.2:** A simple modular lattice of length  $n \geq 3$  contains a sublattice isomorphic to the lattice of figure 2.4 ( $M_{3,3}$ ) or figure 2.5.



**Figure 2.4**  $M_{3,3}$

having a homomorphic image isomorphic to  $M_{3,3}$ .

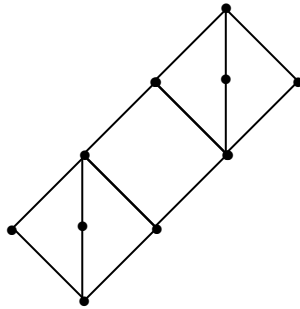


Figure 2.5  $M_{3,3}$

**Proof:** Let  $P_k$  be the class mentioned in corollary 2.5 which has direct links with two or more classes then the two terms which the class  $P_k$  takes would give sublattices isomorphic to  $M_{3,3}$  or to figure 2.5, that is a sublattice with a homomorphic image isomorphic to the lattice  $M_{3,3}$ .

**Corollary 2.6:**  $L$  is a simple modular lattice of length  $n$ , with  $n \geq 4$  then either there are  $(n-2)$  different sublattices of the type mentioned in the above lemma or there exists a sublattice in  $L$  isomorphic to the lattice of figure 2.6, figure 2.7, figure 2.8 or figure 2.9, that is a sublattice with a homomorphic image isomorphic to the lattice  $M_{3,3,3}$ .

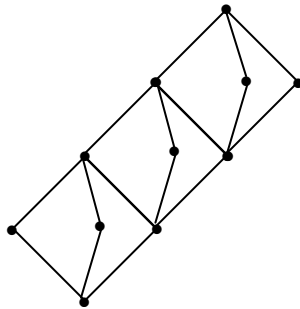
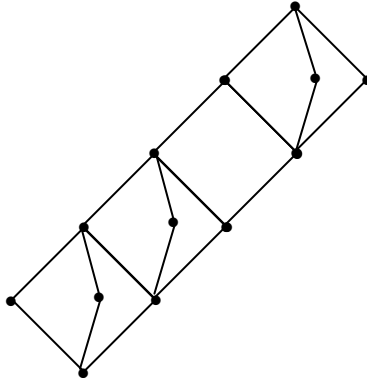
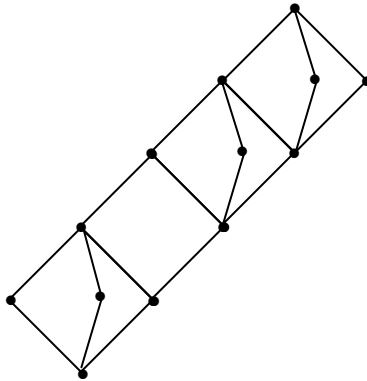


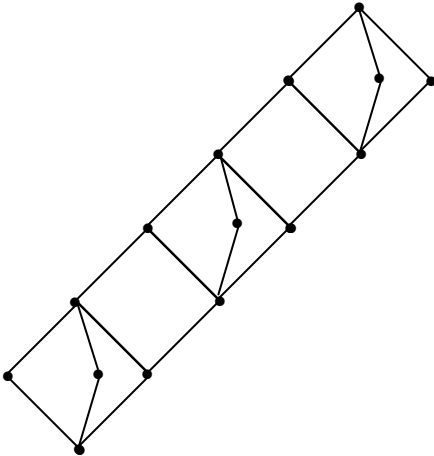
Figure 2.6  $M_{3,3,3}$



**Figure 2.7**  $M_{3,3,3}$



**Figure 2.8**  $M_{3,3,3}$



**Figure 2.9**  $M_{3,3,3}$

## Chapter Three

# SUPERMODULAR LATTICES

Distributive lattices and modular lattices are the two well known equational classes of lattices. In this chapter we introduce another equational class of lattices - called the supermodular lattices. This equational class lies between the equational class of modular lattices and the equational class of distributive lattices.

It is well known that a modular lattice is nondistributive if and only if it contains a sublattice isomorphic to  $M_3$ . In a similar fashion, we prove that a modular lattice is nonsupermodular if and only if it contains a sublattice whose homomorphic image is isomorphic to  $M_4$  or  $M_{3,3}$ .

Further we obtain (cf. Theorem 3.6). A super modular lattice is isomorphic to a subdirect union of copies of  $C_2$  and  $M_3$ .

**DEFINITION 3.1:** A lattice  $L$  is said to be supermodular if it satisfies the following identity

$(a + b) (a + c) (a + d) = a + bc (a + d) + cd (a + b) + bd (a + c)$  for all  $a, b, c, d$  in  $L$ .



**Lemma 3.1:** Every supermodular lattice is modular.

**Proof:** Put  $c = d$

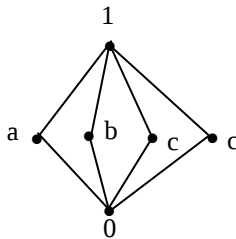
$$\begin{aligned}(a+b)(a+c) &= a + bc + c(a+b) + bc \\ &= a + c(a+b)\end{aligned}$$

is true for all  $a, b, c$  in  $L$ , which can easily be recognized as the modular law.

**Lemma 3.2:** Every modular lattice is not necessarily supermodular.

**Proof:** By an example.

Consider the elements  $a, b, c, d$  as marked in the lattice  $M_4$  of figure 3.1 then



**Figure 3.1**  $M_4$

$$\begin{aligned}(a+b)(a+c)(a+d) &= 1 \\ a+bc(a+d)+cd(a+b)+db(a+c) &= a.\end{aligned}$$

Hence it is not supermodular.

**Lemma 3.3:** Every distributive lattice is supermodular.

**Proof:** If  $L$  is distributive then

$$L.H.S. = R.H.S. = a + bcd.$$

**Lemma 3.4:** In a modular lattice  $L$  if  $a, b, c, d$  are 4 elements such that any two are comparable then this set of 4 elements satisfies the supermodular law.

**Proof:** Let  $a > b$ .

$$(a + b)(a + c)(a + d) = a \geq a + bc(a + d) + bd(a + c) + cd(a + b) \geq a.$$

So the law holds.

Let  $a < b$

$$\begin{aligned} \text{L.H.S.} &= b(a + c)(a + d) = b(a + c(a + d)) \\ &= a + bc(a + d). \end{aligned}$$

$$\text{So } a + bc(a + d) \geq \text{R.H.S.} \geq a + bc(a + d).$$

$$\text{So L.H.S.} = \text{R.H.S.}$$

Let  $b > c$

$$\begin{aligned} (a + b)(a + c)(a + d) &= (a + c)(a + d) \\ &= a + c(a + d) \\ &= a + bc(a + d). \end{aligned}$$

So

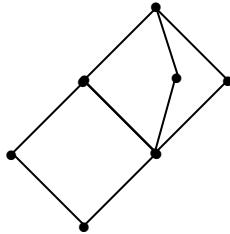
$$\begin{aligned} \text{L.H.S.} &= a + bc(a + d) \geq \text{R.H.S.} \\ &\geq a + bc(a + d). \end{aligned}$$

So the law is satisfied.

**Lemma 3.5:** Every supermodular lattice is not necessarily distributive.

**Proof:** By an example.

The lattice of figure 3.2 is supermodular but not distributive.



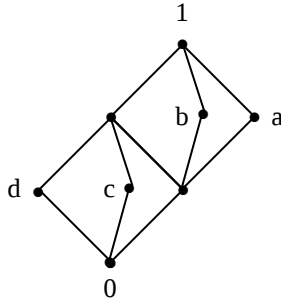
**Figure 3.2**

Supermodularity of  $L$  can easily be checked as  $L$  is a modular lattice and does not contain elements  $a, b, c, d$  such that any two of these are mutually incomparable (cf. Lemma 3.4).

**Lemma 3.6:** The lattices  $M_4$  and  $M_{3,3}$  are non-supermodular.

**Proof:**  $M_4$  is already shown to be non supermodular in Lemma 3.2.

$M_{3,3}$  is not supermodular. For consider the elements  $a, b, c, d$  as shown in  $M_{3,3}$  of figure 3.3 then  $(a + b)(a + c)(a + d) = 1$  while  $a + bc(a + d) + cd(a + b) + db(a + c) = a$ .



**Figure 3.3**  $M_{3,3}$

**Lemma 3.7:** Direct sum of supermodular lattices is supermodular.

**Proof:** Straight forward.

**Lemma 3.8:** Any sublattice of a supermodular lattice is supermodular.

**Proof:** Obvious.

**Lemma 3.9:** Any homomorphic image of a supermodular lattice is supermodular.

**Proof:** Follows as the supermodular identity is a finite identity.

**THEOREM 3.1:** *The class of supermodular lattices is an equational class of lattices lying between the equational class of modular lattices and the equational class of distributive lattices.*

**Proof:** Follows from Lemma 3.1, Lemma 3.3, Lemma 3.7, Lemma 3.8 and Lemma 3.9.

**Lemma 3.10:** If  $L$  is a supermodular lattice then no homomorphic image of  $L$  is isomorphic to  $M_4$  and  $M_{3,3}$ .

**Lemma 3.11:** If  $L$  is a lattice which is not supermodular then  $L$  contains a set of 4 elements  $a, b_1, c_1, d_1$  such that  $(a + b_1)(a + c_1)(a + d_1) > a$  while  $a > b_1 c_1 (a + d_1)$ ,  $c_1 d_1 (a + b_1)$ ,  $d_1 b_1 (a + c_1)$  holds.

$(b_1, c_1, d_1)$  being distinct  $b_1 \neq c_1$  otherwise  $b_1 = b_1 c_1$  and  $a + b_1 = a + b_1 c_1$ , a contradiction as it will imply equality of  $(a + b_1)(a + c_1)(a + d_1) = a$ .

**Proof:** Let  $L$  be a modular lattice which is not supermodular. As  $L$  is not supermodular there exist elements  $p, x, y, z$  such that

$$(p + x)(p + y)(p + z) \underset{\neq}{>} p + xy(p + z) + yz(p + x) + zx(p + y).$$

$$\text{Put } a = p + xy(p + z) + yz(p + x) + xz(p + y)$$

$$b_1 = x$$

$$c_1 = y$$

$$d_1 = z.$$

$$\begin{aligned} \text{Then } a + b_1 &= p + xy(p + z) + yz(p + x) + xz(p + y) + x \\ &= p + x + yz(p + x) \\ &= p + x. \end{aligned}$$

$$\text{Similarly } a + c_1 = p + y \text{ and } a + d_1 = p + z.$$

$$\text{So } (a + b_1)(a + c_1)(a + d_1) \underset{\neq}{>} a.$$

**Lemma 3.12:** If  $L$  is a modular lattice which is not supermodular then  $L$  contains a set of 4 distinct elements  $a, b, c, d$  such that

$$a + b = a + c = a + d \underset{\neq}{>} a.$$

Further  $a > bc, cd, db$  holds.

**Proof:** As  $L$  is not supermodular, by Lemma 3.11 we can assert the existence of a set of 4 elements  $a_1, b_1, c_1, d_1$  in  $L$  such that

$$(a + b_1) (a + c_1) (a + d_1) \underset{\neq}{>} a$$

and

$$\begin{aligned} a &> b_1 c_1 (a + d_1) \\ c_1 d_1 (a + d_1) \end{aligned}$$

and

$$b_1 d_1 (a + c_1)$$

Put

$$\begin{aligned} b &= b_1 (a + c_1) (a + d_1) \\ c &= c_1 (a + b_1) (a + d_1) \\ d &= d_1 (a + b_1) (a + c_1) \end{aligned}$$

then

$$\begin{aligned} a + b &= a + b_1 (a + c_1) (a + d_1) \\ &= (a + b_1) (a + c_1) (a + d_1) \end{aligned}$$

as  $(a + c_1) (a + d_1) > a$  and  $L$  is modular.

Similarly

$$a + c = (a + b_1) (a + c_1) (a + d_1)$$

and

$$a + d = (a + b_1) (a + c_1) (a + d_1).$$

Therefore

$$a + b = a + c + a + d \underset{\neq}{>} a.$$

Now as

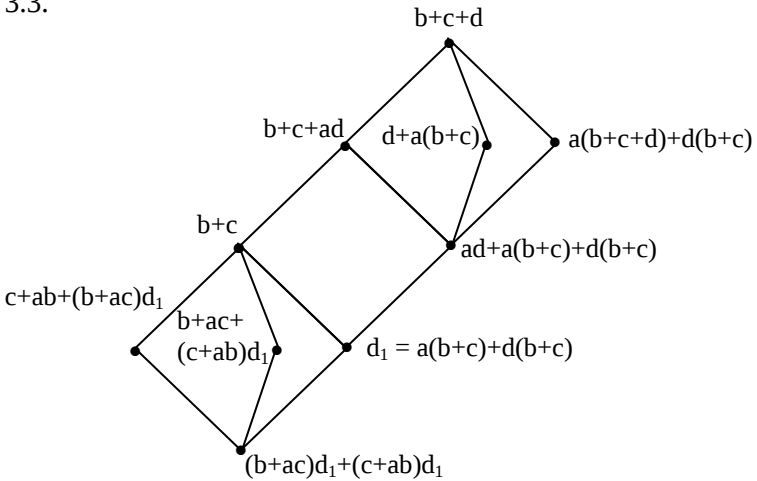
$$\begin{aligned} bc &= b_1 (a + c_1) (a + d_1) c_1 (a + b_1) (a + d_1) \\ &= b_1 c_1 (a + d_1). \end{aligned}$$

We get  $a > bc$ . Similarly  $a > cd$  and  $db$ .

**Lemma 3.13:** If  $L$  is a modular lattice which is non supermodular then for the set of 4 elements  $a, b, c, d$  of Lemma 3.12, all the three lattices generated by  $(a, b, c)$ ,  $(a, c, d)$  and  $(a, d, b)$  are non-distributive.

**Proof:** Let if possible the lattice generated by  $(a, b, c)$  be a distributive lattice; then  $a + b = (a + b)(a + c) = a + bc = a$  (as  $a > bc$ ); a contradiction. Similarly we can prove the nondistributive nature of the other two lattices.

**Lemma 3.14:** If  $L$  is a modular lattice which is non supermodular then for the set of 4 elements  $a, b, c, d$  of Lemma 3.13, the sublattices as shown in figures 3.4, 3.5, and 3.6 have homomorphic images isomorphic to the lattice  $M_{3,3}$  of figure 3.3.



**Figure 3.4**

**Proof:** As  $a + b = a + c = a + d$  we have  
 $b + c + d < a + b$   
 $[a + c + ad] + [d + a(b + c)] = b + c + d$   
 $[b + c + ad] + [a(b + c + d) + d(b + c)]$   
 $= [[b + c + ad] + a(b + c + d)] + d(b + c)$   
 $= (b + c + d)[a + b + c + ad] + d(b + c)$  ( $L$  is modular)  
 $= (b + c + d) + d(b + c)$  (as  $a + b > b + c + d$ )  
 $= b + c + d.$   
 $[d + a(b + c)] + [a(b + c + d) + d(b + c)]$   
 $= d + a(b + c + d)$   
 $= (d + a)(b + c + d)$  ( $L$  is modular)  
 $= b + c + d$  (as  $a + d > b + c + d$ ).

So much for the unions for the first diamond. Next

$$\begin{aligned}
 & [b + c + ad] [d + a (b + c)] \\
 = & ad + (b + c) [d + a(b + c)] \\
 & \quad (L \text{ is modular and } ad < d + a (b + c)). \\
 = & ad + d (b + c) + a (b + c) \\
 & \quad (L \text{ is modular and } b + c > a (b + c)).
 \end{aligned}$$

$$\begin{aligned}
 & [b + c + ad] [a (b + c + d) + d (b + c)] \\
 = & d (b + c) + [b + c + ad] [a (b + c + d)] \\
 & \quad (L \text{ is modular, } d (b + c) < b + c + ad) \\
 = & d (b + c) + (b + c + d) (a (b + c) + ad) \\
 & \quad (L \text{ is modular and } a > ad) \\
 = & d (b + c) + a (b + c) + ad \\
 & \quad (as (b + c + d) > a (b + c) + ad). \\
 & [d + a (b + c)] [a (b + c + d) + d (b + c)] \\
 = & d (b + c) + [d + a (b + c)] [a (b + c + d)] \\
 & \quad (L \text{ is modular and } d(b+c), d + a(b+c)) \\
 = & d (b+c) + [ad + a (b+c)] [b + c + d] \\
 & \quad (L \text{ is modular and } a > a (b + c)) \\
 = & d (b + c) + ad + a (b + c) (as ad + a (b + c) < (b + c + d)).
 \end{aligned}$$

Thus the verification for the upper diamond is complete.

As for the middle square

$$\begin{aligned}
 & (b + c) + [ad + a (b + c) + d(b + c)] \\
 = & b + c + ad. \\
 & (b + c) [ad + a (b + c) + d (b + c)] \\
 = & a (b + c) + d (b + c) + ad (b + c) \\
 & \quad (as L \text{ modular and } (b + c) > a (b + c) + d (b + c)) \\
 = & a (b + c) + d (b + c).
 \end{aligned}$$

Next we come to the lower diamond.

$$\begin{aligned}
 & [(b + ac) + (c + ab)d_1] + d_1 \\
 = & (b + ac) + d_1 \\
 = & (b + ac) + a (b + c) + d (b + c) \\
 = & (b + c) (a + b + ac) + d (b + c) \\
 & \quad (L \text{ is modular and } (b + c) > b + ac). \\
 = & (b + c) + d (b + c) \\
 & \quad [as a + b > b + c \text{ by Lemma 3.12}] \\
 = & b + c.
 \end{aligned}$$

$$\begin{aligned}
& [(c + ab) + (b + ac)d_1] + d_1 \\
&= (c + ab) + d_1 \\
&= (c + ab) + a(b + c) + d(b + c) \\
&= (b + c)(a + c + ab) + d(b + c) \\
&\quad \text{[L is modular and } (b + c) > c + ab\text{]} \\
&= b + c + d(b + c) \\
&\quad \text{[as } a + c > b + c \text{ by Lemma 3.12]} \\
&= b + c \\
&\quad [(b + ac) + (c + ab)d_1] + [(c + ab) + (b + ac)d_1] \\
&= b + c.
\end{aligned}$$

Next for the intersections in the lower diamond we have

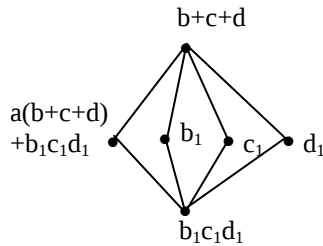
$$\begin{aligned}
& d_1 [(b + ac) + (c + ac)d_1] \\
&= (b + ac)d_1 + (c + ab)d_1 \quad \text{[L is modular]} \\
& d_1 [(c + ab) + (b + ac)d_1] \\
&= (c + ab)d_1 + (b + ac)d_1 \quad \text{[L is modular].} \\
& [(b + ac) + (c + ab)d_1] [(c + ab) + (b + ac)d_1] \\
&= (c + ab)d_1 + (b + ac) [(c + ab) + (b + ac)d_1] \\
&\quad \text{[L is modular and } (c + ab)d_1 < c + ab + (b + ac)d_1\text{]} \\
&= (c + ab)d_1 + (b + ac)d_1 + b(a + ac)(c + ab) \\
&\quad \text{[L is modular and } (b + ac)d_1 < (b + ac)\text{]} \\
&= (c + ab)d_1 + (b + ac)d_1 + ac + b(c + ab) \\
&\quad \text{[L is modular and } ac < c + ab\text{]} \\
&= (c + ab)d_1 + (b + ac)d_1 + ac + bc + ab \\
&\quad \text{[L is modular and } b > ab\text{]} \\
&= (c + ab)d_1 + (b + ac)d_1 + ab + ac \\
&\quad \text{[as } a > bc \text{ implies } ab > bc\text{]} \\
&= (c + ab)d_1 + (b + ac)d_1 \\
&\quad [(b + ac)d_1 > ac, (c + ab)d_1 > ab].
\end{aligned}$$

Thus the proof is complete for figure 3.4. The proof in the case of figures 3.5 and 3.6 follows from symmetry in the figures and as there exists perfect symmetry in the elements  $b, c, d$ . Now the homomorphism which annuls  $(b + c, b + c + ad)$ ,  $(d_1, d_1 + ad)$  gives the homomorphic image isomorphic to the lattice  $M_{3,3}$  of figure 3.3.

**Lemma 3.15:** If  $L$  is a modular lattice which is non supermodular then for the four elements  $a, b, c, d$  of Lemma

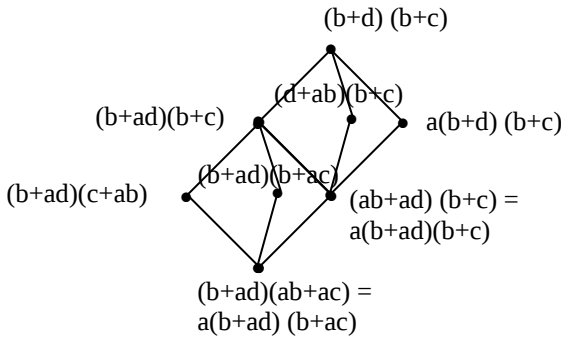


3.14 in the case  $b + c + d = b + c + ad = c + d + ab = d + b + ac$ ; there exists either a sublattice isomorphic to the lattice  $M_4$  as shown in figure 3.7.



**Figure 3.7**

or a sublattice isomorphic to  $M_{3,3}$  as shown in figures 3.8 and 3.9.



**Figure 3.8**

**Proof:** Given  $b + c + ad = b + c + d = c + d + ab = d + b + ac$ .

Consider

$$b_1 = b + ac + ad$$

$$c_1 = c + ab + ad$$

$$d_1 = d + ab + ac$$

then  $b_1 + c_1 = c_1 + d_1 = d_1 + b_1 = b + c + d$ . Also  $a + b_1 = a + b = a + c_1 = a + d_1$ .

Let  $a_1 = a (b_1 + c_1)$   
then

$$\begin{aligned} a_1 + b_1 &= a (b_1 + c_1) + b_1 \\ &= (a + b_1) (b_1 + c_1) \\ &= (b_1 + c_1) [as a + b_1 = a + b > b_1 + c_1]. \end{aligned}$$

Similarly

$$\begin{aligned} a_1 + c_1 &= b_1 + c_1 \\ a_1 + d_1 &= a (c_1 + d_1) + d_1 \\ &= (a + d_1) (c_1 + d_1) \\ &= c_1 + d_1 \\ &= b_1 + c_1. \end{aligned}$$

Hence the sum of any two of  $a_1, b_1, c_1, d_1$  equals  $b + c + d$ .

Next

$$\begin{aligned} a_1 b_1 &= a (b_1 + c_1) b_1 \\ &= ab_1 \\ &= a (b + bc + ad) \\ &= ab + ac + ad. \end{aligned}$$

Similarly,

$$\begin{aligned} a_1 c_1 &= a_1 d_1 = ab + ac + ad \\ b_1 c_1 &= (b + ac + ad) (c + ab + ad) \\ &= ac + ad + b (c + ab + ad) \\ &\quad (L \text{ is modular and } ac + ad < (c + ab + ad)) \\ &= ab + ac + ad + b (c + ad). \end{aligned}$$

Therefore

$$b_1 c_1 \geq ab + ac + ad.$$

Similarly

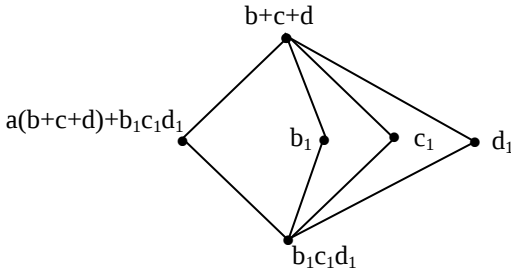
$$c_1 d_1 > ab + ac + ad$$

and

$$d_1 b_1 > ab + ac + ad.$$

Now two cases arise

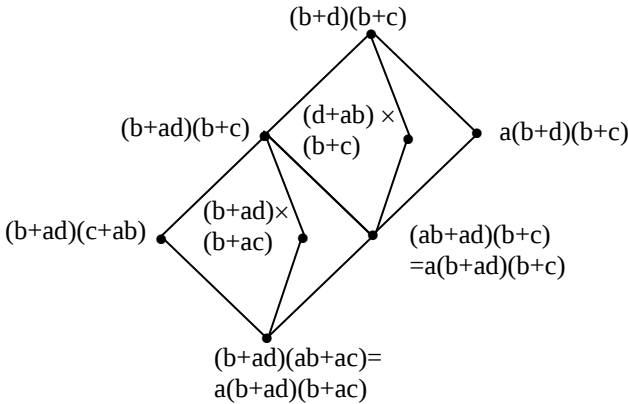
- (1) Either  $b_1 c_1 = c_1 d_1 = d_1 b_1 = b_1 c_1 d_1$  or one of  $b_1 c_1 \neq b_1 c_1 d_1$ .  
In the first case we have the lattice as shown in figure 3.7 isomorphic to the lattice  $M_4$  consisting of  $b + c + d$ ,  $a_2 = (a_1 + b_1 c_1 d_1)$ ,  $b_1, c_1, d_1$  and  $b_1 c_1 d_1$ .



**Figure 3.7**

If further the elements  $b + ac + ad$ ,  $c + ab + ad$ ,  $ab + ac + d$ ,  $b + c + d$  coincide, then the sublattice isomorphic to  $M_4$  collapses.

In this situation the elements  $(b + d)(b + c)$ ,  $a(b+d)(b+c)$ ,  $(d+ab)(b+c)$ ,  $(b + ad)(b+c)$ ,  $(ab+ad)(b+c)$ ,  $(b+ad)(b+ac)$ ,  $(b+ad)(c+ab)(c+ab)$  and  $(b+ad)(ab+ac)$  form a sublattice isomorphic to  $M_{3,3}$  of the lattice as shown in figure 3.8.



**Figure 3.8**

$$\begin{aligned}
 &\text{For a } (b+d)(b+c) + (d+ab)(b+c) \\
 &= (b+d)(b+c)[a(b+c) + d + ab] \\
 &\quad (b+c+d \geq a(b+c) + d + ab) \\
 &\quad \geq ab + ac + d = b + c + d) \\
 &= (b+d)(b+c)(b+c+d) \\
 &= (b+d)(b+c).
 \end{aligned}$$

$$\begin{aligned}
& a(b+d)(b+c) + (b+ad)(b+c) \\
&= (b+c)(b+d)(a(b+c) + b + ad) \\
&= (b+c)(b+d)(b+c+d) \\
&= (b+c)(b+d).
\end{aligned}$$

$$\begin{aligned}
& (b+c)(d+ab) + (b+c)(b+ad) \\
&= (b+c)(d+ab+(b+c)(b+ad)) \\
&\quad (as \ ab < (b+c)(b+ad)) \\
&= (b+c)(d+b+c(b+ad)) \\
&\quad (as \ c(b+ad) < (b+ad)b+d) \\
&= (b+c)(b+d). \\
&\quad (b+c)(b+ad)a(b+c)(b+d) \\
&= (b+c)a(b+ad) = (b+c)(ab+ad).
\end{aligned}$$

$$\begin{aligned}
\text{Next } & (b+c)(d+ab)a(b+c)(b+d) = (b+c)a(d+ab) \\
&= (b+c)(ab+ad).
\end{aligned}$$

$$\begin{aligned}
\text{Also } & (b+c)(b+ad)(b+c)(d+ab) \\
&= (b+c)(ab+(b+ad)d) \\
&= (b+c)(ab+bd+ad) \\
&= (b+c)(ab+bd+ad) \quad (a > bd \text{ so } ad > bad) \\
&= (b+c)(ab+ad).
\end{aligned}$$

Now for the lower diamond

$$\begin{aligned}
& (b+c)(ab+ad) + (b+ac)(b+ad) \\
&= ab + ad(b+c) + b + ad(b+ac) \\
&= b + ad(b+c) = (b+ad)(b+c).
\end{aligned}$$

$$\begin{aligned}
& (b+c)(ab+ad) + (b+ad)(c+ab) \\
&= ab + ad(b+c) + ab + c(b+ad) \\
&= ab + ad(b+c) + ab + c(b+ad) \\
&= ab + (b+ad)(ad(b+c) + c) \\
&= ab + (b+ad)(ad+c)(b+c) \\
&= (b+ad)(b+c)(ab+ad+c) \\
&= (b+ad)(b+c)(b+c+d) \\
&\quad (as \ c + ab + ad = b + c + d) \\
&= (b+ad)(b+c).
\end{aligned}$$

$$\begin{aligned}
& (b+ac) (b + ad) + (b+ad) (c+ab) \\
&= b + ad (b+ac) + ab + c(b+ad) \\
&= b + (b+ad) (c+ad (b+ac)) \\
&= (b+ad) (b+c).
\end{aligned}$$

As for intersections

$$\begin{aligned}
& (b+c) (ab+ad) (b+ac) (b+ad) \\
&= (b+ac) (ab+ad) = a (b+ad) (b+ac) \\
& (b+c) (ab+ad) (b+ad) (c+ab) \\
&= a (b+ad) (c+ab) = a(b+ad) (b+ac).
\end{aligned}$$

$$\begin{aligned}
& (b+ad) (b+ac) (b+ad) (c+ab) \\
&= (b+ad) (b+ac) (c+ab) \\
&= (b+ad) (ab+c(b+ac)) \\
&= (b+ad) (ab+cb+ac) \\
&= (b+ad) (ab+ac) \quad (\text{as } a > bc, ab > bc) \\
&= a (b+ad) (b+ac).
\end{aligned}$$

Further  $M_{3,3}$  cannot collapse, as this will mean

$$a(b+d) (b+c) = (b+d) (b+c)$$

implies

$$a(b+d) (b+c) + a(b+d) = (b+d) (b+c) + a(b+d)$$

that is

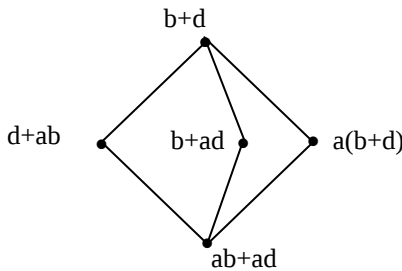
$$a(b+d) = (b+d) (b+c+a(b+d))$$

$$= (b+d) (b+c+d)$$

$$\text{as } b+c+d = b+c+ad < b+c+a (b+d) < (b+c+d)$$

that is  $a(b+d) = (b+d)$ ;

a contradiction as  $(a, b, d)$  generates a non-distributive lattice, which means the sublattice of the figure 3.9 exists.



**Figure 3.9**

(2) Next let us consider the alternate possibility. That is among  $b_1c_1, c_1d_1, d_1b_1$  atleast one is different from  $b_1c_1d_1$ . Without loss in generality let it be  $b_1c_1$ , that is  $b_1c_1 > b_1c_1d_1$ .

Then the elements  $(b_1+d_2) (c_1+d_2) c_1(b_1+d_2), b_1(c_1+d_2), (a_1+b_1c_1) (d_1+b_1c_1) = d_2, (a_1+b_1c_1d_1) (d_1+b_1c_1), d_1 (a_1+b_1c_1), b_1c_1$  and  $b_1c_1d_1$  form a sublattice isomorphic to the lattice of figure 3.10 For

$$\begin{aligned} & b_1c_1 [(a_1 + b_1c_1d_1) (d_1+b_1c_1)] \\ &= b_1c_1 (a_1 + b_1c_1d_1) \\ &= a_1b_1c_1 + b_1c_1d_1 \text{ (L is modular)} \\ &= b_1c_1d_1 \text{ (as } b_1c_1d_1 > a_1b_1c_1). \end{aligned}$$

Also  $b_1c_1 [d_1 (a_1+b_1c_1)] = b_1c_1d_1$ .

Next

$$\begin{aligned} & [d_1 (a_1+b_1c_1)] [(a_1+b_1c_1d_1) (d_1+b_1c_1)] \\ &= d_1 (a_1+b_1c_1d_1) \\ &= a_1d_1 + b_1c_1d_1 \text{ (L is modular)} \\ &= b_1c_1d_1 \text{ (as } a_1d_1 < b_1c_1d_1). \end{aligned}$$

Now for unions for the lower diamond

$$\begin{aligned} & b_1c_1 + [(d_1 + b_1c_1) (a_1+b_1c_1d_1)] \\ &= (d_1 + b_1c_1) [b_1c_1 + a_1 + b_1c_1d_1] \text{ (L is modular)} \\ &= (d_1 + b_1c_1) (a_1 + b_1c_1) \\ &= d_2. \end{aligned}$$

$$\begin{aligned} & b_1c_1 + [d_1 (a_1 + b_1c_1)] \\ &= (b_1c_1 + d_1) (a_1 + b_1c_1) \text{ (L is modular)} \\ &= d_2. \end{aligned}$$

$$\begin{aligned} & [(a_1 + b_1c_1d_1) (d_1 + b_1c_1)] + [d_1 (a_1 + b_1c_1)] \\ &= (d_1 + b_1c_1) [a_1 + b_1c_1d_1 + d_2 (a_1 + b_1c_1)] \\ & \quad \text{(L is modular)} \\ &= (d_1 + b_1c_1) [a_1 + d_1 (a_1 + b_1c_1)] \\ &= (d_1 + b_1c_1) [(a_1 + d_1) (a_1 + b_1c_1)] \\ &= (d_1 + b_1c_1) (a_1 + b_1c_1) \text{ (as } a_1+d_1 = a_1 + b_1). \end{aligned}$$

For the intersections in the upper diamond

$$\begin{aligned}
[b_1 (c_1+d_2)] [c_1 (b_1+d_2)] &= b_1 c_1 \\
[b_1 (c_1 + d_2)] d_2 &= b_1 d_2 \\
&= b_1 (a_1 + b_1 c_1) (d_1 + b_1 c_1) \\
&= (a_1 b_1 + b_1 c_1) (d_1 + b_1 c_1) \\
&= b_1 c_1 (d_1 + b_1 c_1) \quad (\text{as } a_1 b_1 < b_1 c_1) \\
&= b_1 c_1.
\end{aligned}$$

Similarly

$$c_1 d_2 = b_1 c_1.$$

As regard unions

$$b_1 (c_1+d_2) + d_2 = (b_1 + d_2) (c_1 + d_2).$$

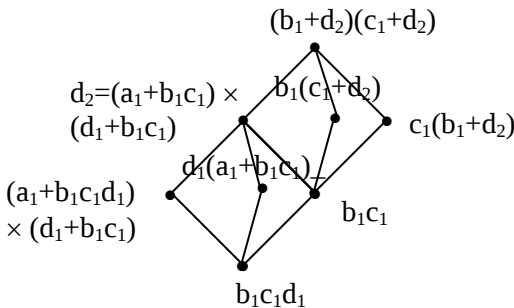
Similarly

$$c_1 (b_1 + d_2) + d_2 = (c_1 + d_2) (b_1 + d_2).$$

Also

$$\begin{aligned}
&b_1 (c_1+d_2) + c_1 (b_1 + d_2) \\
&= (b_1 + d_2) [b_1 (c_1 + d_2) + c_1] \\
&= (b_1 + d_2) [(b_1 + c_1) (c_1 + d_2)] \\
&= (b_1 + d_2) (c_1 + d_2) \quad (\text{as } b + c_1 > b_1 + d_2).
\end{aligned}$$

Thus the sublattice of figure 3.10



**Figure 3.10**

exist. No two elements of this lattice can equal as this will imply the equality of  $b_1 c_1$  and  $b_1 c_1 d_1$  - a contradiction. Thus the proof of Lemma 3.15 is complete.

Combining these lemmas we obtain.

**THEOREM 3.2:** *If  $L$  is a modular lattice having a set of 4 elements  $a, b, c, d$  satisfying the following*

- (1)  $a + b = a + c = a + d \neq a$
- (2)  $a > bc, cd, db$ .

*Then either  $L$  has a sublattice isomorphic to  $M_4$  or contains a sublattice whose homomorphic image is isomorphic to  $M_{3,3}$ .*

Theorem 3.2 with Lemma 3.12 gives.

**THEOREM 3.3:** *A modular lattice  $L$  is supermodular if and only if  $L$  has no sublattice whose homomorphic image is isomorphic to  $M_4$  and  $M_{3,3}$ .*

**Corollary 3.1:** *A lattice is supermodular if and only if  $L$  does not contain a pertagon or sublattice whose homomorphic images are isomorphic to  $M_4$  and  $M_{3,3}$ .*

**Lemma 3.16:** *The dual of a lattice  $L$  is supermodular if and only if there exists a set of four elements  $a, b, c, d$  in  $L$  such that  $ab = ac = ad < a$ ; and  $a < b + c, c + d, d + b$ .*

**Proof:** Easy verification

As the dual of a modular lattice is modular and the lattices  $M_4$  and  $M_{3,3}$  are self dual.

By duality, we obtain

**THEOREM 3.4:** *A modular lattice  $L$  is dually supermodular if and only if  $L$  has no sublattice isomorphic to  $M_4$  and has no sublattice whose homomorphic image is isomorphic to  $M_{3,3}$ .*

As an immediate corollary we get.

**THEOREM 3.5** *A lattice  $L$  satisfies*

*$(a + b)(a + c)(a + d) = a + bc(a + d) + cd(a + b) + db(a + c)$  for all  $a, b, c, d$  in  $L$  if and only if it satisfies  $ab + ac$*



$+ ad = a(b + c + ad)(c + d + ab)(a + b + ac)$  for all  $a, b, c, d$  in  $L$ .

**Proof:** The proof follows as any supermodular or dually supermodular lattice is modular.

Next we obtain a characterization of the subdirectly irreducible supermodular lattices using a result due to B. Johnson [1968] [cf. Theorem of preliminaries) combined with a result due to G. Grätzer (1966).

“A subdirectly irreducible modular lattice of length  $n \geq 3$  contains a sublattice whose homomorphic image is isomorphic to  $M_{3,3}$ .

**THEOREM 3.6:** *A subdirectly irreducible supermodular lattice is isomorphic to the two element chain  $C_2$  or the five element modular lattice  $M_3$ .*

**Proof:** Let  $L$  be a subdirectly irreducible lattice. If length of  $L$  is 1 then  $L$  is isomorphic to  $C_2$ . If length of  $L$  is 2 then as  $L$  is subdirectly irreducible it can either be  $M_3$  or has a sublattice isomorphic to the lattice  $M_4$ .

But  $L$  is supermodular and hence the other possibilities cease to exist. Hence  $L$  is isomorphic to  $M_3$ .

If length of  $L \geq 3$  then as  $L$  is subdirectly irreducible and modular (being supermodular), we can apply B. Johnson’s result [ ] and obtain the sublattice isomorphic to  $M_{3,3}$  which in turn gives rise to a contradiction as then  $L$  ceases to be supermodular. Thus the proof of the theorem.

Combining this with the famous Birkhoff’s theorem we obtain.

**Theorem 3.7:** *A lattice  $L$  is supermodular if and only if  $L$  is a subdirect union of two element chain and the five element modular lattice  $M_3$ .*

**Proof:** If  $L$  is supermodular, then by Birkhoff's theorem it is a subdirect union of subdirectly irreducible supermodular lattices. Using the above we see that any subdirectly irreducible supermodular lattice is either  $C_2$  or  $M_3$ .

Hence  $L$  is a subdirect union of copies of  $C_2$  and  $M_3$ .

For the converse it will suffice to see that both  $C_2$  and  $M_3$  are supermodular. Also by lemma 3.7 and 3.8 direct union of supermodular lattices is supermodular and sublattice of a supermodular lattice is supermodular. Hence subdirect union of supermodular lattices is supermodular. So a subdirect union of copies of  $C_2$  and  $M_3$  is supermodular.

**Corollary 3.1:** If a lattice  $L$  is supermodular then every element of  $L$  has atmost two relative complements in any interval.

**Proof:** Follows from Theorem 3.7.

**Note:** The converse of Corollary 3.2 is not necessarily true.

**Proof:** By an example.

The lattice  $M_{3,3}$  satisfies the requirements that every element of  $L$  has atmost two relative complements in any interval but  $L$  is obviously not supermodular.

**Corollary 3.2:** Let  $L$  be a supermodular lattice and let  $I = (x_1, x_2)$  be a prime interval in  $L$ . Let  $I_1 = (y_1, y_2)$  and  $I_2 = (z_1, z_2)$  be any two intervals of  $L$  such that  $I, I_1$  and  $I_2$ , belong to a chain of  $L$ . If  $I_1$  and  $I_2$  are annulled by  $\theta_1$  (the congruence generated by  $I$ ) then atleast one of  $I_1, I_2$  is a trivial interval.

**Proof:** It is a direct consequence of Theorem 3.6.

**Remark 3:** The primeness of the interval  $I = (x_1, y_1)$  is absolutely essential in the above corollary. Otherwise  $\theta_1$  can be written as a nontrivial sum of two other congruence and hence  $I$

may have more than an interval as its lattice translate belonging to a chain of  $L$ .

(For example the interval  $I$  of the supermodular lattice  $L$  of the adjoining figure 3.11).

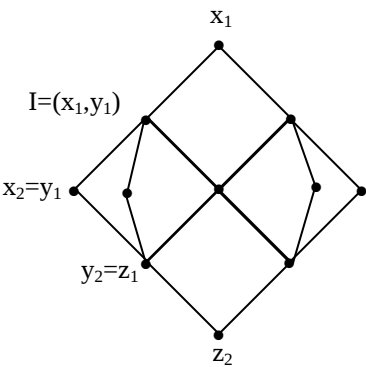


Figure 3.11

**Remark 3.2:** The converse of Corollary 3.3 is not necessarily true.

**Proof:** By an example

Consider  $M_4$  which satisfies the hypothesis for the converse of Corollary 3.3. Nevertheless it is not supermodular.

**Corollary 3.4:** The congruence generated by a prime interval is separable in any supermodular lattice.

**Proof:** It is a direct consequence of Corollary 3.3.

## Chapter Four

# SEMI-SUPERMODULAR LATTICES

In the last chapter we studied supermodular lattices. An equational class of modular lattices generated by the finite modular lattice  $M_3$ . In this chapter, we initiate a study into a series of equational classes of modular lattices termed  $n$ -semi-supermodular lattices, for each finite integer  $n$ . The classes of distributive and supermodular lattices correspond to the integers  $n = 2, 3$  respectively. For  $n \geq 4$ , we observe that these equational classes are no longer generated by their finite members. Further we show that a lattice  $L$  is  $n$ -semi-supermodular if and only if it does not contain any sublattice whose homomorphic image is isomorphic to  $M_{i_1+2, i_2+2, \dots, i_k+2}$  and  $\hat{M}_{i_1+2, i_2+2, \dots, i_k+2}$  such that  $i_1, i_2, \dots, i_k$  are integers  $\geq 1$  with  $i_1 + i_2 + \dots + i_k = (n-1)$ .

To avoid cumber some calculations we give rigorous proof in the case of  $n = 4$  and indicate that the method of proof adopted for  $n = 4$  can be extended for any general  $n$  (finite).

We start with the definition of 4-semi-supermodularity in lattices, which we choose to call semi-supermodular leaving the integer 4 for convenience.

**DEFINITION 4.1:** *A lattice  $L$  is said to be semi-supermodular if it satisfies the following identity.*

$(a+x_1) (a+x_2) (a+x_3) (a+x_4) = a + x_1x_2 (a + x_3) (a + x_4) + x_1x_3 (a + x_2) (a + x_4) + x_1x_4 (a + x_2) (a+x_3) + x_2x_3 (a+x_1) (a+x_4) + x_2x_4 (a+x_1) (a+x_3) + x_3x_4 (a+x_1) (a+x_2)$  for all  $a, x_1, x_2, x_3, x_4$  in  $L$ .

**Lemma 4.1** Any semi-supermodular lattice is modular.

**Proof:** Put  $x_2 = x_3 = x_4$  we get the modular law.

**THEOREM 4.1:** *A modular lattice  $L$  is not semi-supermodular if and only if  $L$  contains elements  $a, x_1, x_2, x_3, x_4$  such that  $a + x_1 = a+x_2 = a + x_3 = a + x_4 > a$  and  $a > x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4$  and  $x_3x_4$ .*

**Proof:** If  $L$  contains elements  $a, x_1, x_2, x_3, x_4$  as specified in the lemma then for this set of 5 elements of  $L$  the left hand side equals  $a + x_1$  while the right hand side equals  $a$  and the two are distinct.

Hence  $L$  is not semi-supermodular.

Conversely if  $L$  is not semi-supermodular then there exist a set of 5 elements  $a, x_1, x_2, x_3, x_4$  in  $L$  such that semi-supermodular identity is not satisfied in  $L$ .

Let

$$\begin{aligned} x_1 &= x_1 (a + x_2) (a+x_3) (a+x_4) \\ x_2 &= (a + x_1) x_2 (a+x_3) (a+x_4) \\ x_3 &= (a + x_1) (a + x_2) x_3 (a+x_4) \\ x_4 &= (a + x_1) (a+x_2) (a+x_3) x_4. \end{aligned}$$

$A = \text{R.H.S.}$  of the semi-supermodular identity when  $a, x_1, x_2, x_3, x_4$  are substituted in it. The set of 5 elements  $A, x_1, x_2, x_3$  and  $x_4$  satisfy the requirements of the lemma.

As a corollary we get.

**Corollary 4.1:** Any supermodular lattice is semi-supermodular.

**Proof:** Equivalently we can prove if a lattice is not semi-supermodular then it is not supermodular. This follows as a consequence of the last lemma and lemma of Chapter 2 which states a lattice is not supermodular if and only if it contains a set of 4 elements  $a, x_1, x_2, x_3$  such that  $a + x_1 = a + x_2 = a + x_3 \geq a$  and  $a > x_1x_2, x_1x_3$  and  $x_2x_3$ .

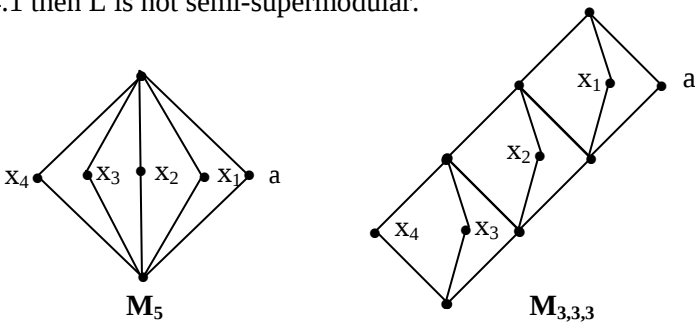
**Lemma 4.2:** Any semi-supermodular lattice is not necessarily supermodular.

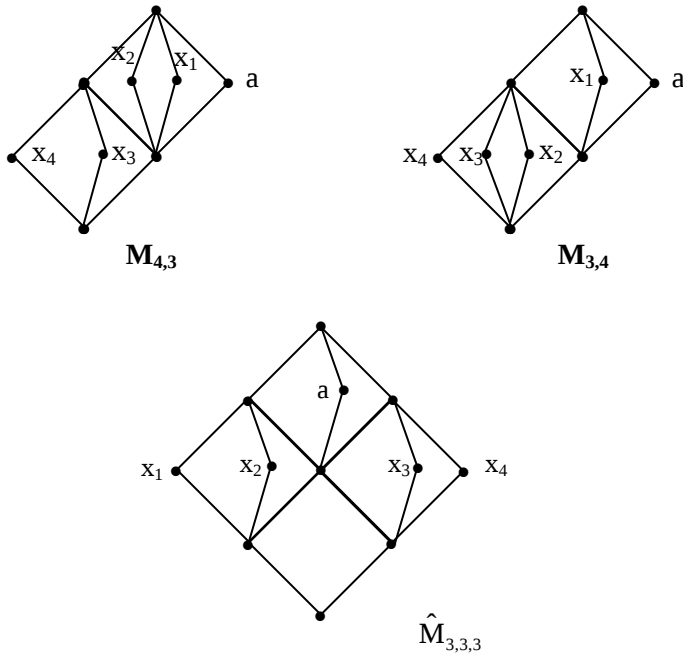
**Proof:** By examples.

These lattices  $M_4$  and  $M_{3,3}$  are semi-supermodular but not supermodular.

Thus the equational class of semi-supermodular lattice lies between the equational class of modular lattices and the equational class of supermodular lattices.

**Lemma 4.3:** If a lattice  $L$  contains a sublattice whose homomorphic image is isomorphic to the lattices of the figures 4.1 then  $L$  is not semi-supermodular.





**Figure 4.1**

**Proof:** The set of 5 elements as marked in the figures satisfy the requirements of theorem 4.1 and hence these cannot be got as homomorphic images of sublattices of semi-supermodular lattices.

Now let  $L$  be a modular, non semi-supermodular lattice, then by Theorem 4.1,  $L$  contains a set of 5 elements  $a, x_1, x_2, x_3, x_4$  such that

$$a + x_1 = a + x_2 = a + x_3 = a + x_4 > a$$

with  $a > x_i x_j$  ( $i \neq j$  for all  $i, j = 1, 2, 3, 4$ )

Consider the set  $T$  of elements given by

$$x_1 + x_2 + x_3 + ax_4$$

$$x_1 + x_2 + ax_3 + x_4$$

$$x_1 + ax_2 + x_3 + x_4$$

$$ax_1 + x_2 + x_3 + x_4.$$

Two cases may arise;

- I. At least one of the elements in  $T$  is different from  $x_1 + x_2 + x_3 + x_4$ .
- II. Every member of  $T$  equals  $x_1 + x_2 + x_3 + x_4$ .

If case I occurs, let us without loss in generality assume that one of the elements different from  $x_1 + x_2 + x_3 + x_4$  is  $x_1 + x_2 + x_3 + ax_4$ .

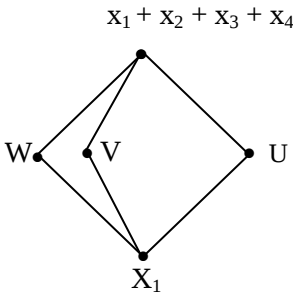
Now as  $x_1 + x_2 + x_3 + ax_4 < x_1 + x_2 + x_3 + x_4$ . We have a sublattice isomorphic to  $M_3$  consisting of elements.

$$x_1 + x_2 + x_3 + x_4, \quad U = a(x_1 + x_2 + x_3 + x_4)$$

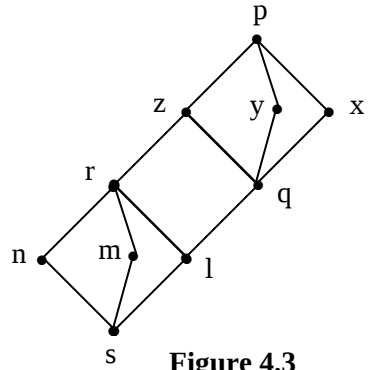
$$x_4 (x_1 + x_2 + x_3) \quad V = a(x_1 + x_2 + x_3) + x_4$$

$$W = x_1 + x_2 + x_3 + ax_4 \text{ and}$$

$X_1 = a(x_1 + x_2 + x_3) + ax_4 + x_4 (x_1 + x_2 + x_3)$  as shown in figure 4.2.



**Figure 4.2**



**Figure 4.3**

Now the set  $a, x_1, x_2, x_3$  satisfies all the properties required for theorem 4.1. In case it has a sublattice isomorphic to  $M_{3,3}$  of figures 3.4, 3.5, 3.6 of chapter III then let  $S$  be renamed as shown in the figures 4.3 then  $p < x_1 + x_2 + x_3 + ax_4$  and not less than or equal to  $a(x_1 + x_2 + x_3) + ax_4 + x_4(x_1 + x_2 + x_3) = X_1$ . Put  $X_1 + x = K_1$  then  $K_i$  lies between  $X_1$  and  $x_1 + x_2 + x_3 + ax_4$  and  $X = K_1p$  lies between  $x$  and  $p$ . Let  $Y = y + X_z$  and  $Z = z + X_y$  then  $p, X, Y, Z, X_y + X_z$  is isomorphic to  $M_3$ . Also  $l_1 = r(X_y + X_z)$  lies between  $l$  and  $r$ . Let  $m_1 = l_1n + m$  and  $n_1 = l_1m + n$  then



$r, l_1, m_1, n_1, l_1m + l_1m$  is isomorphic to  $M_3$ . Also the elements  $(K_1 + U) (K_1 + V), (K_1 + U), (K_1+V), W + (K_1 + U) (K_1 + V), (x_1 + x_2 + x_3 + x_4)$  form a sublattice isomorphic to  $M_3$ . These three sublattices  $M_3$  combine to give us to required  $M_{3,3,3}$  (cf. figure 4.4)

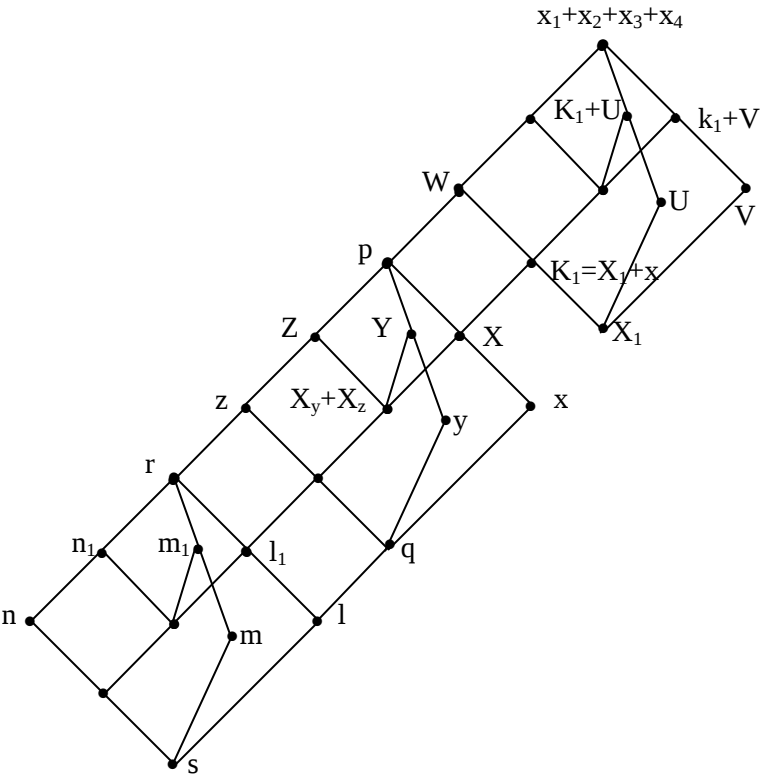


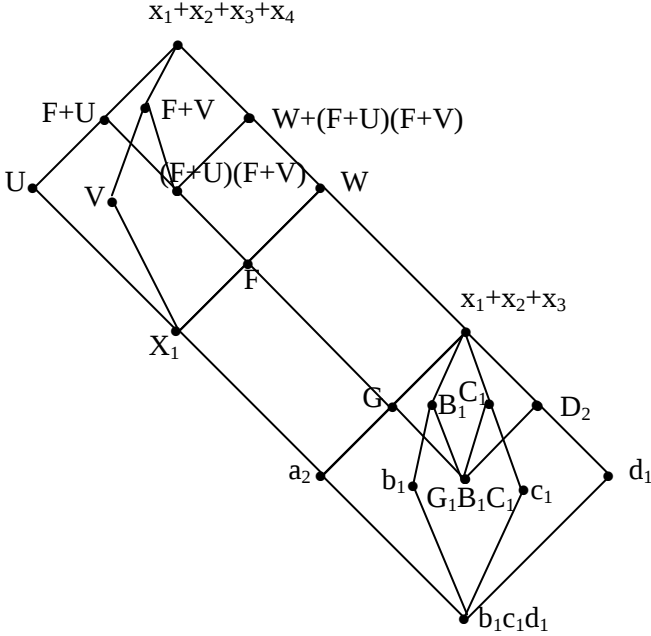
Figure 4.4

The only difficulty we may face is when  $K_1 = x_1 + x_2 + x_3 + ax_4$ . But this could be avoided by choosing a  $> x_3 (x_1 + x_2)$  in which case  $K_1$  will coincide with  $x_1$ ; thus avoiding the difficulty.

Next if  $a, x_1, x_2, x_3$  satisfy the condition  $x_1 + x_2 + ax_3 = x_1 + ax_2 + x_3 = ax_1 + x_2 + x_3 = x_1 + x_2 + x_3$ ; then let  $b_1 = x_1 + ax_2 +$

$ax_3$ ;  $c_1 = ax_1 + x_2 + ax_3$ ;  $d_1 = ax_1 + ax_2 + x_3$  and  $a_1 = a(x_1 + x_2 + x_3)$ .

In case (1)  $b_1c_1 = c_1d_1 = d_1b_1 = b_1c_1d_1$  then by taking  $a_2 = a_1 + b_1c_1d_1$  the elements  $x_1 + x_2 + x_3$ ,  $a_2$ ,  $b_1$ ,  $c_1d_1$ ,  $b_1d_1c_1$  form a sublattice isomorphic to  $M_4$  as shown in figure 4.5.



**Figure 4.5**

In this case, consider  $F = X_1 + b_1c_1d_1$  and  

$$\begin{aligned} F(x_1 + x_2 + x_3) &= a_2 + x_4(x_1 + x_2 + x_3) \\ &= G. \end{aligned}$$

Let

$$\begin{aligned} B_1 &= b_1 + Gd_1 \\ C_1 &= c_1 + Gd_1 \\ D_1 &= d_1. \end{aligned}$$

These satisfy

$$\begin{aligned} G + B_1 &= G + C_1 = G + D_1 = B_1 + D_1 = C_1 + D_1 \\ D_1G &= D_1B_1 = D_1C_1 = d_1G. \end{aligned}$$

We have two possibilities

(1) If  $GB_1 = GC_1 = B_1C_1 = GB_1C_1$  then  $D_2 = D_1 + GB_1C_1$  together with  $G, B_1, C_1, x_1 + x_2 + x_3$  and  $GB_1C_1$  gives a sublattice isomorphic to  $M_4$ . Also as  $F$  lies between  $X_1$  and  $W$ .

$$(F + U) (F + V), (F + U), (F + V),$$

$$W + (F + U) (F + V) \text{ and } x_1 + x_2 + x_3 + x_4$$

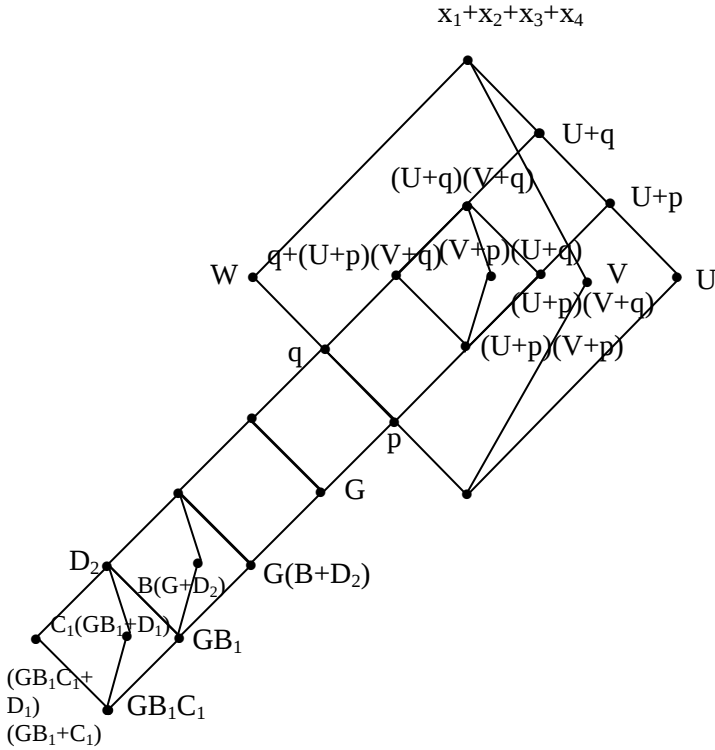
form a sublattice isomorphic to  $M_3$ . These two sublattices combine to give us the sublattice isomorphic to  $M_{3,4}$  as shown in figure 4.5.

(2) In case one of  $GB_1, GC_1, B_1C_1$  is different from  $GB_1C_1$ ; let it be  $GB_1 \neq GB_1C_1$ . Then as shown in lemma of the earlier chapter we obtain a sublattice  $S_1$  isomorphic to  $M_{3,3}$  consisting of  $GB_1C_1, GB_1, C_1(GB_1+D_1), (GB_1C_1+D_1)(GB_1+C_1), D_2 = (GB_1+C_1)(GB_1+D_1) G(B_1+D_2), (G+D_2)B_1, (G+D_2)(B_1+D_2)$ .

Consider the interval  $I = (G, G+D_2)$ ; this interval is projective to  $(G(B_1+D_2), (G+D_2)(B_1+D_2))$  and so is nontrivial, as otherwise  $GB_1 = GB_1C_1$ . Now the interval  $I+F = (F, F+D_2)$  is a nontrivial subinterval of  $J = (F, W)$  as  $J$  is projective to  $(G, x_1+x_2+x_3)$ . Let  $I+F = (p, q)$  (say) then the elements  $(U+p), (V+p), q+(U+p), (V+p), (V+q), (U+q), (V+q)$  form a sublattice isomorphic to  $M_3$ . This together with the sublattice  $S_1$  (mentioned above) gives us the required sublattice with a homomorphic image isomorphic to  $M_{3,3,3}$  as shown in figure 4.6.

If on the other hand one of  $b_1c_1, c_1d_1, d_1b_1$  is other than  $b_1c_1d_1$  then without loss in generality, let us assume (2)  $b_1c_1d_1 \neq b_1c_1$ .

Let  $a_1 = a(x_1+x_2+x_3)$  then  $a_1+b_1 = a_1+c_1 = a_1+d_1 = b_1+c_1 = b_1+d_1 = c_1+d_1$  and  $a_1b_1 = a_1c_1 = a_1d_1$



**Figure 4.6**

Consider the elements  $a_1 + b_1d_1 + c_1d_1$ ,  $b_1 + c_1d_1$ ,  $c_1+d_1b_1$ ,  $d_1$  call them  $a_2$ ,  $b_2$ ,  $c_2$ ,  $d_2$  respectively (say).

Those four elements satisfy the following conditions.

- (1) the sum of any 2 of the four equals a single element  $p$  and
- (2) the product of  $d_1$  with any one of the other three equals a single element  $q$ .

Let  $G = a_2 + x_4 (x_1 + x_2 + x_3)$

$B_1 = b_2 + Gd_2$

$C_1 = c_2 + Gd_2$

$D_1 = d_2$ ,

then we have a situation exactly similar to what we had in the earlier case. A similar reasoning gives us to required result. Thus we have exhausted all the possibilities under case I.

Next we take up the possibilities under case II. Let us recall that under this case every element of the set  $T$  equals  $x_1 + x_2 + x_3 + x_4$ .

Let  $T_1$  be the set of elements

$$T_1 = \left\{ \begin{array}{l} x_1 + x_2 + ax_3 + ax_4; ax_1 + ax_2 + x_3 + x_4 \\ x_1 + ax_2 + x_3 + ax_4; ax_1 + x_2 + ax_3 + x_4 . \\ ax_1 + x_2 + x_3 + ax_4; x_1 + ax_2 + ax_3 + x_4 \end{array} \right.$$

Again two cases may arise.

- (A) Each element of  $T_1$  equals  $x_1 + x_2 + x_3 + x_4$ .
- (B) At least one element of  $T_1$  is different from  $x_1 + x_2 + x_3 + x_4$ .

If (A) is satisfied then set

$$\lambda_1 = x_1 + ax_2 + ax_3 + ax_4$$

$$\lambda_2 = ax_1 + x_2 + ax_3 + ax_4$$

$$\lambda_3 = ax_1 + ax_2 + x_3 + ax_4$$

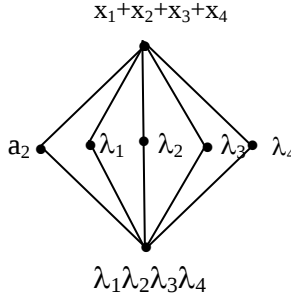
$$\lambda_4 = ax_1 + ax_2 + ax_3 + x_4$$

and  $a_1 = a(x_1 + x_2 + x_3 + x_4)$ .

The sum of any two of the above equal  $x_1 + x_2 + x_3 + x_4$ . The product of  $a_1$  with any one of  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  equal

$$ax_1 + ax_2 + ax_3 + ax_4.$$

If (1)  $\lambda_i \lambda_j = \lambda_1 \lambda_2 \lambda_3 \lambda_4$  for all  $i \neq j$ ,  $i, j = 1, 2, 3, 4$  then the elements  $a_2 (=a_1 + \lambda_1 \lambda_2 \lambda_3 \lambda_4)$ ,  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ ,  $(x_1 + x_2 + x_3 + x_4)$  and  $\lambda_1 \lambda_2 \lambda_3 \lambda_4$  form a sublattice isomorphic to  $M_5$  as shown in figure 4.7.

**Figure 4.7**

If (2)  $\lambda_i \lambda_j = \lambda_j \lambda_k = \lambda_k \lambda_i = \lambda_i \lambda_j \lambda_k$  for some  $i \neq j \neq k$ ,  $(i, j, k) \in (1, 2, 3, 4)$  and  $\lambda_i \lambda_j \lambda_k \neq \lambda_1 \lambda_2 \lambda_3 \lambda_4$  then by a suitable permutation we can assume

$$\lambda_1 \lambda_2 = \lambda_2 \lambda_3 = \lambda_3 \lambda_4 = \lambda_1 \lambda_2 \lambda_3 \text{ and} \\ \lambda_1 \lambda_2 \lambda_3 \neq \lambda_1 \lambda_2 \lambda_3 \lambda_4.$$

Now  $\lambda_1 \lambda_2 \lambda_3 \lambda_4, (a_1 + \lambda_1 \lambda_2 \lambda_3 \lambda_4) (\lambda_1 \lambda_2 \lambda_3 + \lambda_4), \lambda_4 (\lambda_1 \lambda_2 \lambda_3 + a_1) \lambda_1 \lambda_2 \lambda_3, (\lambda_1 \lambda_2 \lambda_3 + a_1) (\lambda_1 \lambda_2 \lambda_3 + \lambda_4)$  form a sublattice isomorphic to  $M_3$ , while the elements  $x_1 + x_2 + x_3 + x_4, a_2 (= a_1 + \lambda_1 \lambda_2 \lambda_3), \lambda_1, \lambda_2, \lambda_3, \lambda_1 \lambda_2 \lambda_3$  form a sublattice isomorphic to  $M_4$ .

Now the side  $(\lambda_1 \lambda_2 \lambda_3, (\lambda_1 \lambda_2 \lambda_3 + a_1), (\lambda_1 \lambda_2 \lambda_3 + \lambda_4))$  of  $M_3$  above is a subinterval of  $(\lambda_1 \lambda_2 \lambda_3, a_2)$  of  $M_4$ .

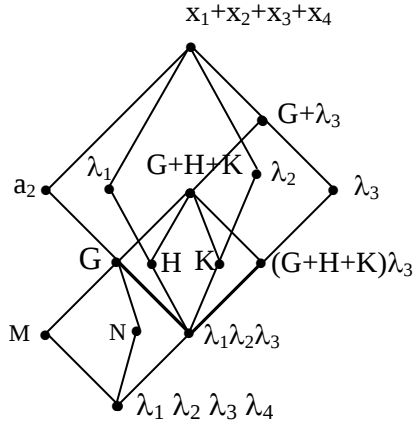
$$\text{Let } G = (\lambda_1 \lambda_2 \lambda_3 + a_1) (\lambda_1 \lambda_2 \lambda_3 + \lambda_4).$$

Consider  $G_1 (G + \lambda_3) \lambda_1, (G + \lambda_3) \lambda_2, \lambda_3$ ; the product of any two of the above four elements equal  $\lambda_1 \lambda_2 \lambda_3$ . The sum of  $\lambda_3$  with any one of the other three equal  $G + \lambda_3$ .

$$\text{Now let } H = (G + \lambda_3) \lambda_1 \text{ and } K = (G + \lambda_3) \lambda_2.$$

In case  $G + H = G + K = H + K = G + H + K$  then  $G + H + K, G, H, K, \lambda_3 (G + H + K), \lambda_1 \lambda_2 \lambda_3$  form a sublattice isomorphic to  $M_4$ ; this combines with the  $M_3$  formed by elements  $\lambda_1 \lambda_2 \lambda_3 \lambda_4,$

$\lambda_4 (\lambda_1 \lambda_2 \lambda_3 + a_1)$ ,  $(a_1 + \lambda_1 \lambda_2 \lambda_3 \lambda_4) (\lambda_1 \lambda_2 \lambda_3 + \lambda_4)$ ,  $\lambda_1 \lambda_2 \lambda_3$ ,  $G$  to give a sublattice isomorphic to  $M_{4,3}$  as shown in figure 4.8.



**Figure 4.8**

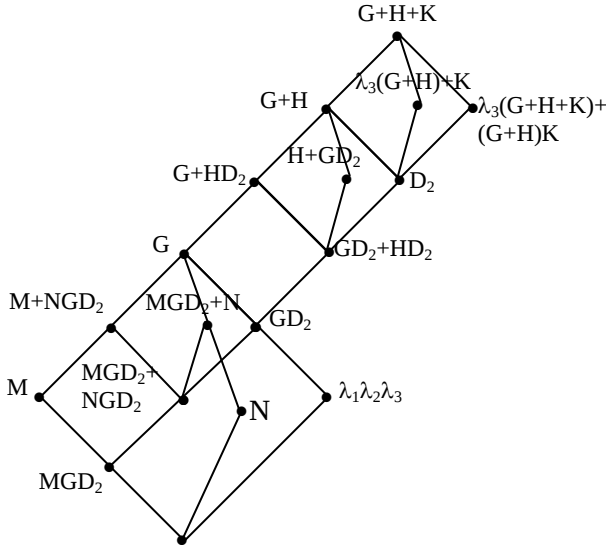
In case one of them say  $G + H \leq G + H + K$  then  $G + H + K$ ,  $(G + H + K) \lambda_3 + (G + H)K$ ,  $\lambda_3(G + H) + K$ ,  $G + H$ ,  $\lambda_3(G + H) + K(G + H) = D_2$ ,  $G + HD_2$ ,  $H + GD_2$ ,  $HD_2 + GD_2$  form a sublattice isomorphic to  $M_{3,3}$ .

Now  $GD_2$  lies between  $G$  and  $\lambda_1 \lambda_2 \lambda_3$ .

Let  $M = \lambda_4 (\lambda_1 \lambda_2 \lambda_3 + a_1)$

$N = (\lambda_1 \lambda_2 \lambda_3 \lambda_4 + a_1) (\lambda_1 \lambda_2 \lambda_3 + \lambda_4)$ .

Then  $G$ ,  $GD_2$ ,  $M + NGD_2$ ,  $N + MGD_2$ ,  $MGD_2 + NGD_2$  form a sublattice isomorphic to  $M_3$ . This sublattice combines with the above mentioned  $M_{3,3}$  to give us a sublattice with a homomorphic image isomorphic to  $M_{3,3,3}$  as shown in figure 4.9.



**Figure 4.9**

Lastly if (iii)  $\lambda_1 \lambda_2 \neq \lambda_1 \lambda_2 \lambda_3 \neq \lambda_1 \lambda_2 \lambda_3 \lambda_4$  then  $\lambda_1 \lambda_2 \lambda_3 \lambda_4$ ,  $M$ ,  $N$ ,  $\lambda_1 \lambda_2 \lambda_3$ ,  $G$  form a sublattice isomorphic to  $M_3$ .

Let

$$M_1 = \lambda_3 (\lambda_1 \lambda_2 + a_1)$$

$$N_1 = (\lambda_1 \lambda_2 \lambda_3 + a_1) (\lambda_1 \lambda_2 + \lambda_3)$$

$$G_1 = (\lambda_1 \lambda_2 + a_1) (\lambda_1 \lambda_2 + \lambda_3)$$

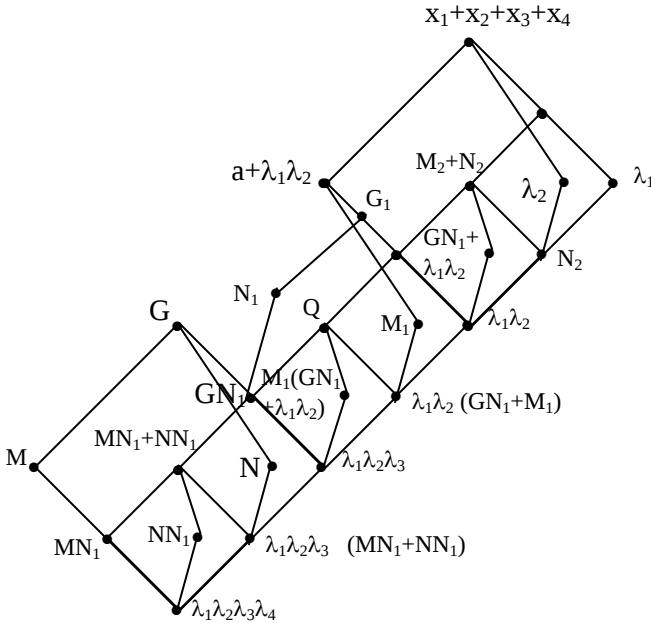
then  $\lambda_1 \lambda_2 \lambda_3$ ,  $M_1$ ,  $N_1$ ,  $\lambda_1 \lambda_2$ ,  $G$  form a sublattice isomorphic to  $M_3$ .

Further  $\lambda_1 \lambda_2$ ,  $\lambda_1$ ,  $\lambda_2$ ,  $a_1 + \lambda_1 \lambda_2$ ,  $x_1+x_2+x_3+x_4$  form a sublattice isomorphic to  $M_3$ .

From these three sublattices we extract a sublattice which has a homomorphic image isomorphic to  $M_{3,3,3}$ . This consists of the elements  $\lambda_1 \lambda_2 \lambda_3 \lambda_4$ ,  $MN_1$ ,  $MN_1$ ,  $\lambda_1 \lambda_2 \lambda_3 (MN_1 + NN_1)$ ,  $MN_1 + NN_1$ ;  $\lambda_1 \lambda_2 \lambda_3$ ,  $GN_1$ ,  $M_1$  ( $GN_1 + \lambda_1 \lambda_2$ ),  $\lambda_1 \lambda_2 = (GN_1 + M_1)$ ,  $Q = (GN_1 + \lambda_1 \lambda_2) (GN_1 + M_1)$ ,



$\lambda_1 \lambda_2, (GN_1 + \lambda_1 \lambda_2), M_2 = \lambda_2 (GN_1 + \lambda_1), N_2 = \lambda_1 (GN_1 + \lambda_2), M_2 + N_2$ . Figure 4.10



**Figure 4.10**

So far the case (A). Next we take up case (B).

Recall that in this every element of  $T$  equals  $x_1 + x_2 + x_3 + x_4$ . In particular  $x_1 + x_2 + x_3 + ax_4 = x_1 + x_2 + x_3 + x_4$ .

This implies

$a(x_1 + x_2 + x_3 + x_4) + x_4 (x_1 + x_2 + x_3) = x_1 + x_2 + x_3 + x_4$  (cf. figure 4.2).

Similarly  $x_1 + x_2 + ax_3 + x_4 = x_1 + x_2 + x_3 + x_4$  implies  $a(x_1 + x_2 + x_3 + x_4) + x_3 (x_1 + x_2 + x_4) = x_1 + x_2 + x_3 + x_4$  and the other two equalities yield

$$a(x_1 + x_2 + x_3 + x_4) + x_2(x_1 + x_3 + x_4) = x_1 + x_2 + x_3 + x_4$$

and

$$a(x_1 + x_2 + x_3 + x_4) + x_1(x_2 + x_3 + x_4) = x_1 + x_2 + x_3 + x_4.$$

Put

$$A = a(x_1 + x_2 + x_3 + x_4)$$

$$X_1 = x_1(x_2 + x_3 + x_4)$$

$$X_2 = x_2(x_1 + x_3 + x_4)$$

$$X_3 = x_3(x_1 + x_2 + x_4)$$

$$X_4 = x_4(x_1 + x_2 + x_3)$$

then  $A + X_1 = A + X_2 = A + X_3 = x_1 + x_2 + x_3 + x_4$ . So  $A + X_i > A$  for all  $i = 1, 2, 3, 4$ .

Now  $X_i X_j = x_i x_j$  hence  $A > X_i X_j$  ( $i \neq j$ ). Also  $X_i + X_j + X_k$   
 $= (x_1 + x_2 + x_3)(x_1 + x_2 + x_4)(x_1 + x_3 + x_4)(x_2 + x_3 + x_4)$   
 $= X_1 + X_2 + X_3 + X_4$

for all  $(i, j, k)$  ( $i \neq j \neq k$ )  $\in (1, 2, 3, 4)$

If for this set of elements  $A, X_1, X_2, X_3, X_4$  the corresponding set  $T_1$  of elements are all equal, then we proceed as in the case (A). If not then without loss in generality let the element of  $T_1$  other than  $X_1 + X_2 + X_3 + X_4$  be  $X_1 + X_2 + AX_3 + AX_4$ .

Either (1)  $A(X_1 + X_2) + X_3 + X_4 \neq X_1 + X_2 + X_3 + X_4$  or

(2)  $A(X_1 + X_2) + X_3 + X_4 = X_1 + X_2 + X_3 + X_4$ .

In case (1) occurs

$$X_1 + X_2 + A(X_3 + X_4) \neq X_1 + X_2 + X_3 + X_4$$

for otherwise the upper diamond of figure 4.11 collapses forcing  $X_1 + X_2 + X_3 + X_4 = A(X_1 + X_2) + X_3 + X_4$  a contradiction. Thus in this case of elements  $X_1 + X_2 + X_3, X_1 + X_2 + A(X_3 + X_4), A(X_1 + X_2) + X_3 + X_4$ ,

$$A(X_1 + X_2 + X_3) + (X_1 + X_2)(X_3 + X_4), A(X_1 + X_2) + A(X_3 + X_4) + (X_1 + X_2)(X_3 + X_4); X_1 + X_2, [A(X_1 + X_2) +$$

$$(X_1 + X_2)(X_3 + X_4) = P, P(X_1 + AX_2) + (X_2 + AX_1), (X_2 + AX_1)P + (X_1 + AX_2),$$

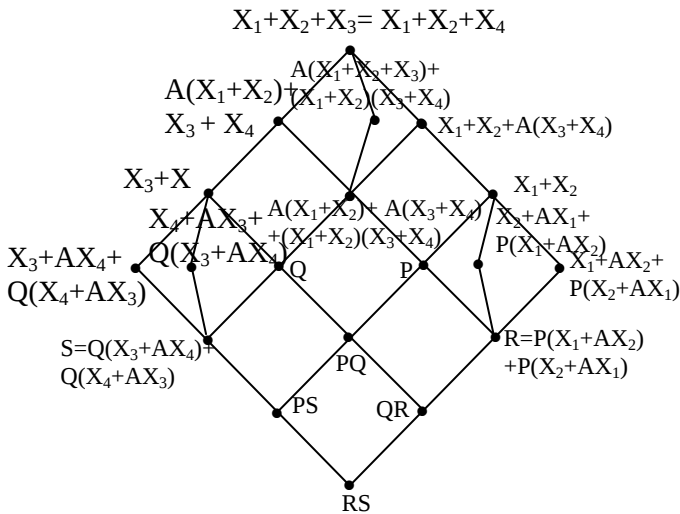
$$R = [P(X_1 + AX_2) + P(X_2 + AX_1)]; X_3 + X_4.$$

$$[A(X_3 + X_4) + (X_1 + X_2)(X_3 + X_4)] = Q,$$

$$(X_3 + AX_4) + Q(X_4 + AX_3), (X_4 + AX_3) + Q(X_3 + AX_4),$$

$$S = [Q(X_4 + AX_3) + Q(X_3 + AX_4)];$$

PQ, PS, QR and RS form a sublattice with a homomorphic image isomorphic to  $\hat{M}_{3,3,3}$  as shown in figure 4.11.



**Figure 4.11**

In the other case that is in (2) we have

$$A(X_1 + X_2) + X_3 + X_4 = X_1 + X_2 + X_3 + X_4.$$

Set

$$A_1 = A(X_1 + X_2 + X_3 + X_4)$$

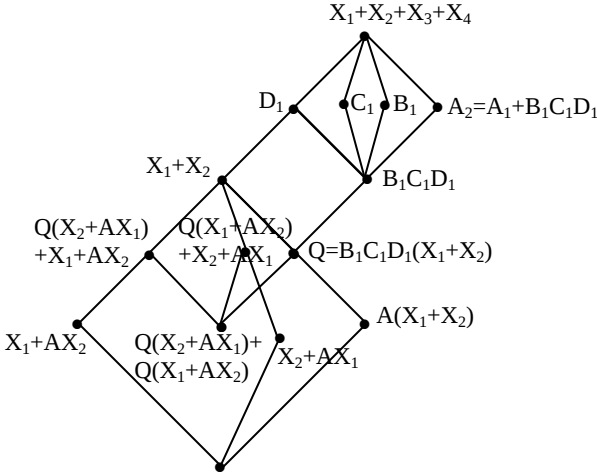
$$B_1 = A(X_1 + X_2) + AX_3 + X_4$$

$$C_1 = A(X_1 + X_2) + X_3 + AX_4$$

$$D_1 = X_1 + X_2 + AX_3 + AX_4.$$

then the sum of any two of the above four elements equals  $X_1 + X_2 + X_3 + X_4$ ; and the product of  $A_1$  with any one of the other three equals  $A(X_1 + X_2) + AX_3 + AX_4$ .

If  $B_1 C_1 = C_1 D_1 = D_1 B_1$  we obtain  $M_{4,3}$  as shown in figure 4.12.



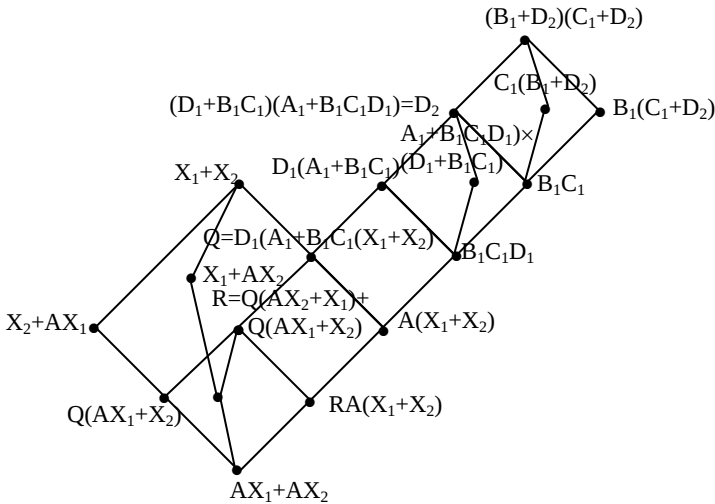
**Figure 4.12**

This consists of the elements  $X_1 + X_2 + X_3 + X_4$ ,  $A_2 = A_1 + B_1 C_1 D_1$ ,  $B_1$ ,  $C_1$ ,  $D_1$ ,  $X_1 + X_2$ ,  $Q = B_1 C_1 D_1 (X_1 + X_2)$ ,  $(X_1 + AX_2) + Q(X_2 + AX_1)$ ,  $Q(X_1 + AX_2) + (X_2 + AX_1)$ ,  $Q(X_1 + AX_2) + Q(X_2 + AX_1)$ .

If  $B_1 C_1 \neq B_1 C_1 D_1$  then we obtain  $M_{3,3,3}$  as shown in figure 4.13. This consists of the elements  $B_1 C_1 D_1$ ,  $D_1(A_1 + B_1 C_1)$ ,  $(A_1 + B_1 C_1 D_1)(D_1 + B_1 C_1)$ ,  $B_1 C_1$ ,  $D_2 = (D_1 + B_1 C_1)(A_1 + B_1 C_1 D_1)$ ,  $C_1(B_1 + D_2)$ ,  $B_1(C_1 + D_2)$ ,  $(B_1 + D_2)(C_1 + D_2)$ .

$R = Q(AX_1 + X_2) + Q(X_1 + AX_2)$  (where  
 $Q = D_1(A_1 + B_1 C_1)(X_1 + X_2)$ ),

$RA(X_1 + X_2)$ ,  $Q(AX_2 + X_1)$ ,  $Q(X_2 + AX_1)$  and  $AX_1 + AX_2$ .



**Figure 4.13**

Thus we conclude this analysis by giving the theorem characterizing semi-supermodular lattices.

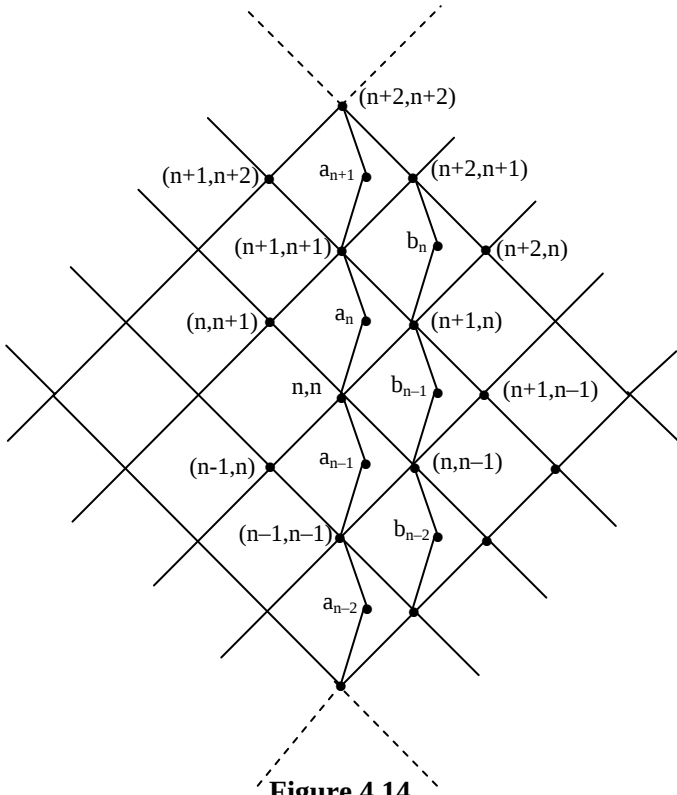
**THEOREM 4.2:** *A modular lattice  $L$  is semi-supermodular if and only if it does not contain any sublattice, whose homomorphic image is isomorphic to  $M_{3,3,3}$ ,  $M_{4,3}$ ,  $M_{3,4}$ ,  $M_5$  or  $\widehat{M}_{3,3,3}$ .*

Dually we observe

**THEOREM 4.3:** *The dual of a modular lattice  $L$  is semi-supermodular if and only if it does not contain any sublattice whose homomorphic image is isomorphic to  $M_{3,3,3}$ ,  $M_{4,3}$ ,  $M_{3,4}$ ,  $M_5$  and  $\widehat{M}_{3,3,3}$  ( $\widehat{M}_{3,3,3}$  is the dual of the lattice  $\widehat{M}_{3,3,3}$ ). (cf. figure 4.1).*

Hence we have.

**THEOREM 4.4:** *The dual of a semi-supermodular lattice  $L$  is not necessarily semi-supermodular.*



**Proof:** By an example. Observe that  $\widetilde{M}_{3,3,3}$  is semi supermodular; while its dual  $\hat{M}_{3,3,3}$  is not.

Unlike the equational class of supermodular lattices, the equational class of semisuper modular contains infinite subdirectly irreducible members.

**THEOREM 4.5:** *The equational class of semi-supermodular lattices is not generated by its finite members.*

**Proof:** It will suffice to give an example of an infinite subdirectly irreducible semi-supermodular lattice.

Consider the lattice  $L$  in figure 4.14.  $L$  consists of the direct product of  $\mathbb{Z}$  (the set of all integers +ve, -ve, 0) with itself with the usual natural direct product order.

For each integer  $n$ , two, extra element  $a_{n-1}$ ,  $b_{n-1}$  are inserted in the squares formed by  $[((n-1), (n-1)), (n, n-1), (n-1, n), (n, n)]$  and  $[(n, n-1), (n+1, n-1), (n, n), (n+1, n)]$  so as to make them sublattices isomorphic to  $M_3$  (see figure 4.14).

Observe that every prime interval of  $L$  is projective to any other prime interval of  $L$ . Thus  $L$  is subdirectly irreducible as it is discrete. Further  $L$  has no sublattice with a homomorphic image isomorphic to  $M_{3,3,3}$ ,  $M_{3,4}$ ,  $M_{4,3}$  or  $M_5$  or  $\widehat{M}_{3,3,3}$ . Hence  $L$  is semi-supermodular.  $L$  is obviously infinite. Hence the statement of the theorem.

Now before we generalize semi-supermodularity for any integer  $n$ , we observe the following facts, which we enunciate in the form of the following lemma.

**Lemma 4.4:** Let  $x_1, x_2, \dots, x_n$  be a set of  $n$  elements ( $n > 3$ ) of a modular lattice  $L$  satisfying

$$\begin{aligned} x_i + x_j &= 1 \text{ for } i \neq j \text{ for all } i, j = 1, 2, \dots, n. \\ x_i x_j &= 0 \text{ for } i \neq j \text{ for all } i, j = 1, 2, \dots, n. \end{aligned}$$

that is they form a sublattice isomorphic to  $M_n$ .

(1) Let  $a_1$  be an element between  $x_1$  and 1, then there exist elements  $a_i$  between  $x_i$  and 1 for all  $i = 2, 3, \dots, n$  such that the sublattice of  $L$  generated by the set of  $n$  elements  $a_1, a_2, \dots, a_n$  contains a homomorphic image isomorphic to

$$M_{i_1+2, i_2+2, \dots, i_r+2} \quad (\text{where } i_1 + i_2 + \dots + i_r = (n-1))$$

$$\text{and } i_1, i_2, \dots, i_r \geq 1).$$

(2) Let  $b_1$  be an element between 0 and  $x_1$ , then there exist element  $b_i$  between 0 and  $x_i$  for each  $i = 2, 3, \dots, n$ , such that the sublattice of  $L$  generated by the set of  $n$  elements  $b_1, b_2, \dots, b_n$  contains a homomorphic image isomorphic to  $M_{i_1+2, i_2+2, \dots, i_r+2}$  (where  $i_1 + i_2 + \dots + i_r = (n-2)$  and  $i_1, i_2, \dots, i_r \geq 1$ ).

Further a subinterval of  $(1, a_1)$  or  $(0, b_1)$  is projective with the essential side of  $M_{i_1+2, i_2+2, \dots, i_r+2}$  so obtained.

**Proposition 4.1:** *Essential side of  $M_{i_1+2, i_2+2, \dots, i_r+2}$  is the side, whose additive or multiplicative translate meets every one of the  $M_i$ 's involved in the figure of  $M_{i_1+2, i_2+2, \dots, i_r+2}$ .*

**Proof:** By mathematical induction on the integer  $n$ . When  $n = 3$ . The elements  $a_1, a_2 = x_2 + a_1x_3, a_3 = x_3 + a_1x_2, a_1x_2 + a_1x_3$  and 1 form a sublattice isomorphic to  $M_3$  with  $(a_1, 1)$  as a side with the elements  $b_1, b_2 = x_2 (a_1 + x_3), b_3 = x_3 (a_1+x_2), (a_1+x_2) (a_1+x_3), 0$  form a sublattice isomorphic to  $M_3$  with  $(0, b_1)$  as a side. Thus when  $n = 3$  the result is true.

Assume the result to be true for all  $n \leq m-1$ . To prove it for  $n = m$ . Set

$$a_2 = a_1x_n + x_2$$

$$a_3 = a_1x_n + x_3$$

.

.

.

$$a_n = x_n$$

then  $a_i + a_j = 1$  for all  $i, j = 1, 2, \dots, n$ ,  $a_ia_n = a_ix_n$  for all  $i = 1, 2, \dots, n-1$ .

(1) if  $a_i a_j = a_1a_2 \dots a_{n-1}$  for all  $i, j = 1, 2, \dots, n-1$  ( $i \neq j$ ) then  $1, a_1, a_2, \dots, a_{n-1}, a_n+a_1a_2 \dots a_{n-1}, a_1a_2, \dots, a_{n-1}$  form a sublattice isomorphic to  $M_n$ .



(2) If  $a_1 a_2 \dots a_{n-1} \neq a_{\sigma_1} \dots a_{\sigma_{n-2}}$  for a permutation  $(\sigma_1, \sigma_2, \dots, \sigma_{n-2}, \sigma_{n-1})$  of  $(1, 2, 3, \dots, n-1)$  then we can rearrange  $1, 2, \dots, n-1$  and get it as  $a_1 a_2 \dots a_{n-1} \neq a_1 a_2 \dots a_{n-2}$ .

If further  $a_i a_j = a_1 a_2 \dots a_{n-2}$  for all  $i, j = 1, 2, \dots, (n-2)$  ( $i \neq j$ ) then we obtain the two sublattices  $M_3$  formed by

$a_1 a_2 \dots a_{n-1}, a_1 a_2 \dots a_{n-2}, c_n = a_{n-1} (a_1 a_2 \dots a_{n-2} + a_n), d_n = (a_1 a_2 \dots a_{n-1} + a_n) (a_1 \dots a_{n-2} + a_{n-1}), (a_1 a_2 \dots a_{n-2} + a_n) (a_1 a_2 \dots a_{n-2} + a_{n-1}) = b_n$  and  $M_{n-1}$  formed by  $1, a_1, a_2, \dots, a_{n-2}, a_n + a_1 a_2 \dots a_{n-2}, a_n + a_1 a_2 \dots a_{n-2}, a_1 a_2 \dots a_{n-2}$ .

Now  $b_n$  lies between  $(a_1 a_2 \dots a_{n-2} + a_n)$  and  $1$ . By induction hypothesis applied to  $b_n$  for this  $M_{n-1}$  we get a sublattice isomorphic to  $M_{i_1+2, i_2+2, \dots, i_k+2}$  with  $i_k > 1$  and  $i_1 + i_2 + \dots + i_k = (n-3)$ .

Further, the essential side of this  $M_{i_1+2, i_2+2, \dots, i_k+2}$  is projective with a subinterval of  $(a_1 a_2 \dots a_{n-2}, b_n)$ ; which helps us to be an adjoining  $M_3$ . Hence we obtain  $M_{i_1+2, i_2+2, \dots, i_k+2, 3}$  with  $i_k \geq 1$  and  $i_1 + \dots + i_k + 1 = (n-2)$  giving us the required result.

(3) if  $a_1 a_2 \dots a_{n-2} \neq a_1 a_2 \dots a_{n-3}$  but  $a_1 a_2 \dots a_{n-3} = a_i a_j$  for all  $i, j = 1, 2, \dots, n-3$ . Set  $e_n = (a_1 a_2 \dots a_{n-2} + a_n) (a_1 \dots a_{n-3} + a_{n-2}) (a_1 a_2 \dots a_{n-2} + a_{n-1})$  then we obtain a  $M_3$  consisting of  $c_n e_n + d_n e_n, c_n e_n, d_n c_n a_1 \dots a_{n-2} (c_n e_n + d_n e_n), a_1 a_2 \dots a_{n-1}$ , another  $M_3$  consisting of  $a_1 a_2 \dots a_{n-2}, e_n (e_n + c_{n-1}) a_1 a_2 \dots a_{n-3}, (e_n + a_1 a_2 \dots a_{n-3}) c_{n-1}, (e_n + c_{n-1}) (e_n + a_1 a_2 \dots a_{n-3})$  such that the side

$c_n e_n + d_n e_n, a_1 \dots a_{n-2} (c_n e_n + d_n e_n)$  of the first  $M_3$  is projective with the side  $(e_n, a_1 \dots a_{n-2})$ .

Also the side  $[(e_n + c_{n-1}) (e_n + a_1 a_2 \dots a_{n-3}), (e_n + c_{n-1}) a_1 \dots a_{n-3}]$  of the lattice  $M_3$  is projective with the interval  $(a_1 a_2 \dots a_{n-3}, e_n + a_1 \dots a_{n-3})$ .

Put  $e_n + a_1 \dots a_{n-3} = f$ ;  $f$  is an element between  $a_1 \dots a_{n-3}$  and  $a_1 a_2 \dots a_{n-3} + a_n$  and the elements  $1, a_1, \dots, a_{n-3}, a_n + a_1 \dots a_{n-3}, a_1 a_2 \dots a_{n-3}$  form a sublattice isomorphic to  $M_{n-2}$ .

Again the induction hypothesis gives us a  $M_{i_1+2, i_2+2, \dots, i_l+2}$  such that  $i_1 + \dots + i_l = (n-4)$ . These  $M_i$ 's together with the  $M_{3,3}$  mentioned above gives us the required  $M$ .

Thus we proceed until we exhaust all the  $a_i$ 's and come to  $a_1 a_2$  alone.

Now we proceed to generalize the notion of semi-supermodularity.

**DEFINITION 4.2:** A lattice  $L$  is called  $n$ -semi supermodular if it satisfies the identity

$$(a+a_1) \dots (a+a_n) \\ = a + \sum_{\substack{i \neq j \\ i, j=1}}^n a_i a_j (a+a_1) \dots (a \hat{+} a_i) (a \hat{+} a_j) \dots (a+a_n)$$

(where  $(a \hat{+} a_i)$  means  $(a+a_i)$  is omitted).

Just as we have proved the results for semi-supermodular lattices we can obtain the following results for any  $n$ -semi supermodular lattices.

**Lemma 4.5:** Any  $n$ -semi-supermodular lattice is modular.

**THEOREM 4.6:** A modular lattice  $L$  is  $n$ -semi-supermodular if and only if there does not exist a set of  $(n+1)$  elements  $a, a_1, \dots, a_n$  in  $L$  such that  $a + a_1 = a + a_2 = \dots = a + a_n > a$  with  $a > a_i a_j$  ( $i \neq j$ ) ( $i, j = 1, 2, \dots, n$ ).

**THEOREM 4.7:** *A modular lattice  $L$  is  $n$ -semi-supermodular if and only if it does not contain a sublattice whose homomorphic image is isomorphic to  $M_{i_1+2, i_2+2, \dots, i_r+2}$  or  $\widehat{M}_{i_1+2, i_2+2, \dots, i_r+2}$  with  $i_1 + i_2 + \dots + i_r = n-1, i, j \geq 1$ .*

**Proof:** We essentially use lemma and adopt a similar method of proof as in the case of Theorem 4.2.

**Corollary 4.1:** The dual of a  $n$ -semi supermodular lattice is not necessarily  $n$ -semi-super modular.

**Corollary 4.2:** Any  $m$ -semi-supermodular lattice is  $n$ -semi-supermodular when  $m \leq n$ . Corollary 4.2 combined with Theorem 4.5 gives;

**Corollary 4.3:** The equational classes of  $n$ -semisupermodular lattices (for  $n \geq 4$ ) are not generated by their finite members.

## Chapter Five

# SOME INTERESTING EQUATIONAL CLASSES OF MODULAR LATTICES

In chapter III, we studied supermodular lattices and characterized them as those modular lattices which do not contain sublattices isomorphic to  $M_4$  or  $M_{3,3}$ . In this chapter we are interested in characterizing those lattices which do not contain sublattices isomorphic to  $M_4$  alone (cf. theorem 5.1 and 5.2). Lastly we define modular elements in a general lattice  $L$  and characterize them in theorem 5.3.

We start with the investigation of the following lattice identity (A).

$$(a+b)(a+c)(a+d)(ab+(a+b)c+d)(ac+(a+c)d+b) \times (ad + (a+d)b+c) = ab + ac + ad + (a+b)(ab+c)d + (a+c)(ac+d)b + (a+d)(ad+b)c \text{ for all } a, b, c, d \text{ in } L.$$

**Remark:** In any lattice the left hand side of equality (A) is in general greater than or equal to the right hand side of (A).

**Lemma 5.1:** Any lattice  $L$  satisfying identity (A) is modular.

**Proof:** Let  $L$  be a lattice satisfying identity (A); put  $b = c$  in (A) we get  $(a+d)(b+ad) = ad + b(a+d)$  for all  $a, b, d$  in  $L$  which can easily be recognized as the modular law. Hence  $L$  is modular. The converse however is not true.

Thus we prove in the following:

**Lemma 5.2:** Every modular lattice need not necessarily satisfy identity (A).

**Proof:** By an example.

Consider the lattice  $L$  of figure 3.1. It does not satisfy the identity (A) for the set  $(a, b, c, d)$  of elements as specified in  $M_4$ . That is, in this case the left hand side of (A) equals 1 and the right hand side equal 0.

As an immediate corollary, we obtain.

**Corollary 5.1:** If  $L$  is any lattice satisfying identity (A) then it cannot contain a sublattice isomorphic to  $M_4$ .

Proof is an easy consequence of the fact that any sublattice  $S$  of a lattice  $L$  satisfying identity (A) also satisfies (A).

Now arises the natural questions whether every modular lattice not satisfying identity (A) contains a sublattice isomorphic to  $M_4$ ?

Fortunately for us, we again have a counter example.

**Lemma 5.3:** A modular lattice  $L$  not satisfying identity (A) need not necessarily contain a sublattice isomorphic to  $M_4$ .

**Proof:** Consider the lattice of figure 5.1.

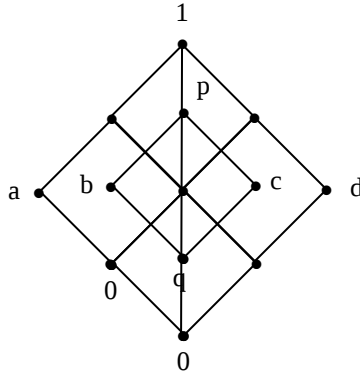


Figure 5.1

and the elements  $a, b, c, d$  as marked in the figure 5.1. The left hand side of identity (A) equals  $p$ , while the right hand side equals  $q$ . Hence  $L$  does not satisfy identity (A). One can easily see from the figure the absence of any sublattice in  $L$ , isomorphic to  $M_4$ .

Next we characterize all modular lattices which satisfy identity (A) in the following theorem.

**THEOREM 5.1:** *A modular lattice  $L$  does not satisfy identity (A) if and only if it contains a non-distributive triple  $(b, c, d)$  such that  $b, c, d$  are the common relative complements of an element 'a' in some interval of  $L$ .*

*[( $b, c, d$ ) is called a non-distributive triple if the lattice generated by the triple is not distributive].*

**Proof:** Let  $L$  be a modular lattice containing a non-distributive triple  $(b, c, d)$  such that  $b, c, d$  are the common relative complements of a single element  $a$  in  $L$  in the interval  $(y, x)$ .

That is

$$\begin{aligned} a+b &= a+c = a+d = x \\ ab &= a.c = a.d = y \text{ and} \\ (b+c) (b+d) (c+d) &> bc+cd+db. \end{aligned}$$

Now the set of four elements  $(a, b, c, d)$  does not satisfy identity (A) in  $L$ . For when these elements are substituted in identity (A) the left hand side equals

$$\begin{aligned} x.x.x (y+xc+d) (y+xd+b) (y+xb+c) \\ = x(c+d) (d+b) (b+c) \quad (\text{as } x > b, c, d > y) \\ = (c+d) (d+b) (b+c) \quad (\text{as } x > b+c). \end{aligned}$$

The right hand side equals

$$\begin{aligned} y + y + y + x (y + c)d + x(y + d)b + x (y + b)c \\ = y + cd + db + bc \quad (\text{as } x > b, c, d > y) \\ = cd + db + bc \quad (\text{as } bc > y). \end{aligned}$$

By hypothesis  $(b, c, d)$  is a non-distributive triple. Hence the two sides are not equal. Conversely let  $L$  be a modular lattice not satisfying identity (A). That is there exists a set of four elements  $(a, b, c, d)$  in  $L$  such that

$$\begin{aligned} (a+b) (a+c) (a+d) (ab+(a+b)c+d) (ac+(a+c)d+b)(ad+(a+d)b+c) \neq \\ ab+ac+ad+(a+b) (ab+c)d + \\ (a+c) (ac+d)b + (a+d) (ad+b)c. \end{aligned} \quad (1)$$

$$\begin{aligned} \text{Let } e_1 &= (a+b) (a+c) (ab+ac+d) \\ &= ab + ac + d (a+b) (a+c) \\ &\quad (L \text{ is modular}) \end{aligned}$$

$$\begin{aligned} e_2 &= (a+c) (a+d) (ac+ad+b) \\ &= ac + ad + b(a+c) (a+d) \end{aligned}$$

$$\begin{aligned} e_3 &= (a+d) (a+b) (ad+ab+c) \\ &= ad+ab+c(a+d) (a+b). \end{aligned}$$

Then  $a + e_1 = a + e_2 = a + e_3 = (a+b) (a+c) (a+d)$  and  
 $ae_1 = ae_2 = ae_3 = ab + ac + ad$ .

$$\begin{aligned}
 e_1 + e_2 &= ab + ac + d(a+b) (a+c) + ac + ad + b(a+c) (a+d) \\
 &= ab + ac + ad + (a+b) (d(a+c) + b(a+c) (a+d)) \\
 &\quad (as (a+b) \geq b(a+c) (a+d)) \\
 &= ab+ac+ad + (a+b) (a+c) (d(a+c) + b (a+d)) \\
 &\quad (as (a+c) \geq d(a+c)) \\
 &= ab+ac+ad+(a+b) (a+c) (a+d) (d(a+c)+d) \\
 &\quad (as (a+d) \geq d(a+c)) \\
 &= (a+b) (a+c) (a+d) (ab+ac+ad+b+d (a+c)) \\
 &\quad (as (a+b) (a+c) (a+d) \geq ab+ac+ad) \\
 &= (a+b) (a+c) (a+d) (ac+d(a+c) + b) \\
 &\quad (as ab < b and ad < (a+c)d).
 \end{aligned}$$

Similarly we obtain

$$\begin{aligned}
 e_2 + e_3 &= (a+b) (a+c) (a+d) (ad+(a+d) b+c) \text{ and} \\
 e_3 + e_1 &= (a+b) (a+c) (a+d) (ab + (a+b) c+d)
 \end{aligned}$$

By symmetry of the operations (+) and (.) in this set up,  
 we have

$$\begin{aligned}
 e_1e_2 &= ab + ac + ad + (a+c) (ac+d)b \\
 e_2e_3 &= ab + ac + ad + (a+d) (ad+b)c \\
 e_3e_1 &= ab + ac + ad + (a+b) (ab+c)d.
 \end{aligned}$$

Hence, the left hand side of ineuqatity (1) equals  
 $(e_1+e_2) (e_2+e_3) (e_3+e_1)$  and the right hand side equals  
 $e_1e_2 + e_2e_3 + e_3e_1$ .

Hence, we establish the existence of a non-distributive triple  
 $(e_1, e_2, e_3)$  in  $L$ , this is turn implies no  $e_i = e_j$  for  $i \neq j$  ( $i, j = 1, 2, 3$ ) as otherwise  $(e_1, e_2, e_3)$  will cease to be non-distributive.



Also  $a \neq e_i$  for any  $i = 1, 2, 3$ ; otherwise if  $a = e_i$  then  $a + e_i = ae_i$ . This implies  $(a+b)(a+c)(a+d) = ab+ac+ad$ .

Now  $(a+b)(a+c)(a+d) \geq e_1, e_2, e_3 \geq ab + ac + ad$ . So the sublattice generated by  $(e_1, e_2, e_3)$  is contained in the convex sublattice.

$$S = (ab + ac + ad, (a+b)(a+c)(a+d))$$

In particular  $(e_1 + e_2)(e_2 + e_3)(e_3 + e_1)$  and  $e_1e_2 + e_2e_3 + e_3e_1$  are elements in  $S$ . If the greatest and the least elements of  $S$  are equal then any two of the elements in the convex sublattice  $S$  are equal. Thus the two elements

$(e_1 + e_2)(e_2 + e_3)(e_3 + e_1)$  and  $e_1e_2 + e_2e_3 + e_3e_1$  are equal.

A contradiction to our hypothesis. Hence we establish the existence of a non-distributive triple  $(e_1, e_2, e_3)$  in  $L$  such that they are the common relative complements of the single element  $a$  in the interval  $(ab + ac + ad, (a+b)(a+c)(a+d))$ .

Lastly we come to the characterization of those modular lattices which contain a sublattice to  $M_4$  in the following.

**THEOREM 5.2:** *A modular lattice  $L$  contains a sublattice isomorphic to  $M_4$  if and only if  $L$  contains a subset of four elements  $(a, b, c, d)$  such that equality (A) is not satisfied by this set. However this set of four elements satisfy the following identity and its dual for every permutation of the element  $(b, c, d)$ .*

$$\begin{aligned} & a(ab+(a+b)c+d)(ac+(a+c)d+b)(ad+(a+d)b+c) + \\ & \quad (a+b)(a+c)(ab+ac+d)(ad+(a+d)b+c) \\ &= (a+b)(a+c)(a+d)(ab+(a+b)c+d) \times (ac+(a+c)d+b) \\ & \quad (ad+(a+d)b+c) \dots (B) \end{aligned}$$

**Proof:** Let  $L$  be a lattice containing a sublattice  $S$  isomorphic to  $M_4$  consisting of  $1, a, b, c, d, 0$ . Then the set of four elements of  $L$  corresponding to the four elements  $(a, b, c, d)$  of  $M_4$  satisfy the requirements of the theorem.

Conversely if  $L$  contains a set of four elements  $(a, b, c, d)$  not satisfying (A) then from the Theorem 5.1 we see that the triple  $(e_1, e_2, e_3)$  is a non-distributive triple, where  $e_1, e_2, e_3$  are as defined in the proof of Theorem 5.1. Further we observe that the equalities mentioned in Theorem 5.2 are precisely the following equalities.

$$\begin{aligned} ap + e_i p &= p(a+q)(e_i+q) = q \text{ for } i = 1, 2, 3 \text{ where} \\ p &= (e_1 + e_2)(e_2+e_3)(e_1+e_3) \text{ and } q = e_1 e_2 + e_2 e_3 + e_3 e_1. \end{aligned}$$

Now  $(q+ap, q+e_1 p, q+e_2 p, q+e_3 p)$  generate a sublattice isomorphic to the lattice of figure 5.1. For

$$\begin{aligned} q + e_1 p &= e_1(e_2 + e_3) + e_2 e_3 \\ q + e_2 p &= e_2(e_1 + e_3) + e_1 e_3 \\ q + e_3 p &= e_3(e_1 + e_2) + e_1 e_2. \end{aligned}$$

So the sum of any two of these is  $p$  and the product of any two of these is  $q$ . Hence they are not equal as  $p$  is not equal to  $q$ .

Also

$$\begin{aligned} q + ap + q + e_i p &= q + ap + e_i p \\ &= q + p = p \end{aligned}$$

and

$$\begin{aligned} (q+ap)(q+e_i p) &= p(q+a)(q+e_i) \\ &= pq = q. \end{aligned}$$

So  $ap + q$  does not equal  $q + e_i p$  for  $i = 1, 2, 3$ .

**DEFINITION 5.1:** *An element  $a$  of a lattice  $L$  is called modular if and only if the sublattice by  $(a, b, c)$  for all  $b, c$  in  $L$  is modular.*

**THEOREM 5.3:** *An element  $a$  of a lattice  $L$  is modular if and only if*

- (1)  $(a+b)(a+c) = a+b(a+c)$  for all  $b, c$  in  $L$ .
- (2)  $ab + ac = a(b+ac)$  for all  $b, c$  in  $L$ .
- (3)  $a+b = a+c; ab = ac; b \geq c$  implies  $b = c$  for all  $b, c$  in  $L$ .
- (4)  $(b+a)(b+c) = b+c(a+b)$  for all  $b, c$  in  $L$ .
- (5)  $ba + bc = b(c+ab)$  for all  $b, c$  in  $L$ .

**Proof:** Let  $a$  be a modular element in a lattice  $L$  and let  $b, c$  be two arbitrary elements of  $L$ . The sublattice  $S$  generated by  $\{a, b, c\}$  is modular (by definition). Further all the elements in equalities (1) to (5) belong to  $S$ . These equalities are satisfied; as  $S$  is modular.

Conversely, let  $a$  be an element of a lattice  $L$  satisfying equalities (1) to (5) and  $b, c$  two arbitrary elements in  $L$ . Consider the sublattice generated by  $S = \{a, b, c\}$  in  $L$ .

First we observe

$$\text{if } x \leq y \text{ in } S \text{ then } (a+x)y = ay+x \quad (6)$$

For if  $p = (a+x)y \leq ay + x = q$  then

$$\begin{aligned} a+p &= a+q = a+x \\ ap &= a+q = a+x \\ ap &= aq = ay \text{ foreing } p = q \text{ by} \end{aligned} \quad (3)$$

As particular cases of (6) we obtain

$$(b+a)(b+c) = b+a(b+c) \quad (4')$$

$$ba + bc = b(a+bc) \quad (5')$$

$$\text{Next } (a+b)(a+c)(b+c) = (a+b(a+c))(b+c) \quad (\text{by (1)})$$

$$= a(b+c) + b(a+c) \quad (\text{by (6)}).$$

Similarly

$$(a + b) (a + c) (b + c) = a(b + c) + c(a + b).$$

$$\text{Next } p = (b+ac) (a+c) \geq (a+c)b + ac = q$$

$$\begin{aligned} a+p &= a+ (b+ac) (a+c) \\ &= (a+b+ac) (a+c) && \text{(by (1))} \\ &= (a+b) (a+c). \end{aligned}$$

$$\begin{aligned} a+q &= a + (a+c)b \\ &= (a+b) (a+c) && \text{(by (1))} \end{aligned}$$

$$\begin{aligned} ap &= a(b+ac) = ab + ac \\ aq &= a((a+c)b+ac) = a(a+c)b + ac \\ &= (ab+ac) && \text{(by (2))} \end{aligned}$$

Therefore  $a+p = a+q$  and  $ap = aq$  forcing  $p=q$  (by (3)).

That is

$$(b+ac) (a+c) = (a+c)b + ac \quad (7)$$

Similarly

$$(c+ab) (a+b) = (a+b)c + ab \quad (8)$$

$$\text{Also } (a+bc) (b+c) = a(b+c) + bc \quad \text{(by (6))}$$

Finally adding (7) and (8) we get

$$(b+ac) (a+c) + (c+ab) (a+b)$$

$$= (a+c)b + (a+b)c$$

$$\begin{aligned} \text{L.H.S.} &= (a+c) (b+ac+(c+ab) (a+b)) \quad \text{(by (4))} \\ &= (a+c) (b+(c+ab) (a+b)) \quad (ac \leq (c+ab) (a+b)) \\ &= (a+c) (b+c (a+b) + ab) \quad \text{(by (8))} \\ &= (a+c) (b+c (a+b)) \\ &= (a+c) (b+c) (a+b) \quad \text{(by (4)).} \end{aligned}$$

$$\text{So } (a+c) (b+c) (a+b) = b(a+c) + c(a+b) \quad (9)$$

The equalities due to (6) - (9) are got dually. Hence the sublattice generated by  $\{a, b, c\}$  in  $L$  is a homomorphic image of the free modular lattice generated by  $(a, b, c)$ ; and hence is modular.

As in the case of neutral elements we can show.

**THEOREM 5.4:** *The set of modular elements of a lattice  $L$  is the intersection of the maximal modular sublattice of  $L$ .*

**Proof:** On the same lines as Birkhoff. 1948.

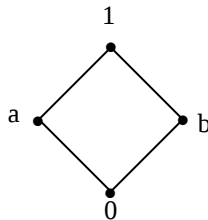
**Corollary 5.2:** The modular elements of any lattice  $L$  form a modular sublattice of  $L$ .

## Chapter Six

# SMARANDACHE LATTICES

In this chapter we for the first time introduce the notion of Smarandache lattices and give a few of its properties. For more about lattices please refer [2].

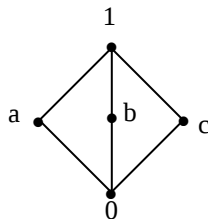
**DEFINITION 6.1:** *Let  $L$  be a lattices; if  $L$  has atleast one sublattice whose homomorphic image is isomorphic to the Boolean algebra of order four (i.e., isomorphic to*



*then we call  $L$  to be a Smarandache lattice (Smar lattice).*

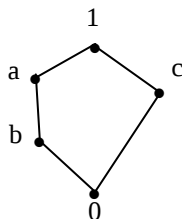
We first illustrate this by some examples.

**Example 6.1:** Let  $L$  be a diamond lattice.



$L$  is a Smar lattice. For the sets  $\{0, 1, a, b\}$ ,  $\{0, 1, a, c\}$  and  $\{1, 0, b, c\}$  are sublattices whose homomorphic image is isomorphic to the Boolean algebra of order four.

**Example 6.2:** Let  $L$  be the pentagon lattice given by



$L$  is a Smar Lattice for the subsets  $\{0, 1, a, c\}$  and  $\{0, 1, b, c\}$  are sublattices of  $L$  whose homomorphic image is isomorphic to the Boolean algebra of order four.

We see the pentagon lattice has also a sublattice of order four given by the set  $\{1, a, b, 0\}$  whose homomorphic image is not isomorphic with the Boolean algebra of order four.

Further both the diamond lattice and pentagon lattice have only three sublattice of order four, but in case of diamond lattice we see all sublattices of order four are isomorphic with the Boolean algebra of order four, however in case of pentagon lattice we see only two of the sublattices are such that their

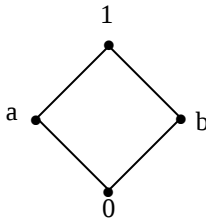
homomorphic image is isomorphic to the Boolean algebra of order four but one sublattice is just a chain lattice of order four.

It is also interesting to note both the pentagon lattice and the diamond lattice are not distributive but both are Smar-lattice.

We see not all lattices are Smar-lattices. Infact we have a class of lattices which are not Smar-lattices.

**THEOREM 6.1:** *Let  $L$  denote the class of all chain lattices; no lattice in  $L$  is a Smar lattice.*

**Proof:** If  $M$  in  $L$  has to be a Smar-lattice we need a proper subset  $S$  in  $M$  of order four such that  $S$  is a sublattice of  $M$  and  $S$  is isomorphic with the Boolean algebra of order four.



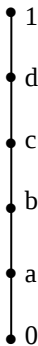
This in turn implies  $S$  in  $L$  should be such that,  $S$  has two distinct elements  $a, b$  other than  $0$  and  $1$  such that  $a + b = 1$  and  $a.b = 0$ .

In a chain lattice  $L$  we know there does not exist distinct elements  $a, b$  ( $a \neq b$ )  $a, b \in L \setminus \{1, 0\}$  such that  $a+b = 1$  and  $a.b = 0$ . So  $S$  cannot be a sublattice whose homomorphic image is isomorphic with the Boolean algebra of order four.

We give some examples before we proceed onto define or discuss more properties about Smar lattices.

**Example 6.3:** Let  $L$  be a chain lattice of order six.  $L$  is not a Smar lattice.

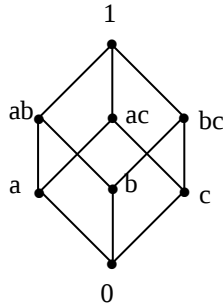




**Example 6.4:** Let  $L$  be a chain lattice of order  $n$ .  $L$  is not a Smar lattice.

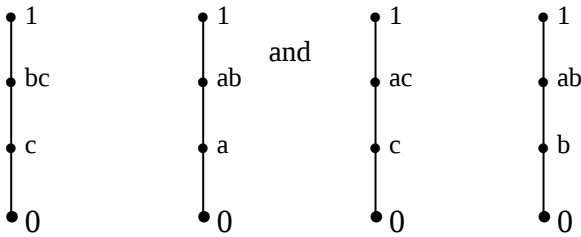


**Example 6.5:** Let  $L$  be a Boolean algebra of order eight given by the following Hasse diagram.

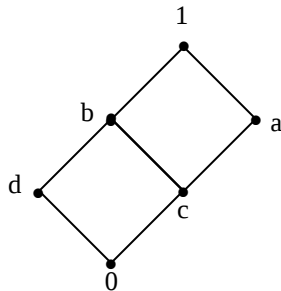


We see this has six sublattices of order four which are such that they are isomorphic with the Boolean algebra of order four. These sublattices are  $\{a, b, 0, ab\}$ ,  $\{0, b, c, bc\}$ ,  $\{ac, bc, c, 1\}$ ,  $\{1, ab, ac, a\}$ ,  $\{0, a, c, ac\}$  and  $\{1, ac, bc, b\}$ .

In fact  $L$  has also sublattices of order four which are chain lattices.



**Example 6.6:**



Let  $L$  be a lattice given by the Hasse diagram.  $L$  is a Smar Lattice for some of the subsets  $S_1 = \{b, c, d, 0\}$ ,  $S_2 = \{1, b, a, c\}$ ,  $S_3 = \{1, d, a, 0\}$ ,  $S_4 = \{1, d, c, 0\}$  are sublattices of  $L$  whose homomorphic image is isomorphic with a Boolean algebra of order four.

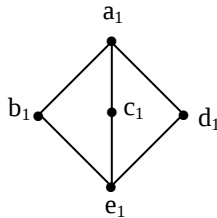
Take  $P_1 = \{1, a, c, 0\}$  and  $P_2 = \{1, b, d, 0\}$  subsets of  $L$ .

These are also sublattices of  $L$  of order four but are not isomorphic with chain lattices or order four.

We now proceed onto prove the following interesting result.

**THEOREM 6.2:** *If  $L$  is a modular lattice then  $L$  is a Smar Lattice.*

**Proof:** Given  $L$  is a modular lattice, so in  $L$  we have triples,  $a, b, c$  such that the modular law is satisfied. Thus  $L$  has sublattices of order five which are isomorphic with the diamond lattice say we have  $S = \{a_1, b_1, c_1, d_1, e_1\} \subseteq L$  such that  $S$  has Hasse diagram,



Now clearly  $\{a_1, b_1, c_1, e_1\}$ ,  $\{a_1, c_1, d_1, e_1\}$  and  $\{a_1, b_1, d_1, e_1\}$  are subsets of  $S$  whose homomorphic image is isomorphic to the Boolean algebra of order four. Thus every modular lattice is a Smar lattice.

Now a natural question would be “Is every distributive lattice a Smar lattice?” This question is partially answered by the following Theorem.

**THEOREM 6.3:** *Every distributive in general is not a Smar lattice.*

**Proof:** We see all chain lattices are distributive. But none of the chain lattice is a Smar lattice. Hence the claim.

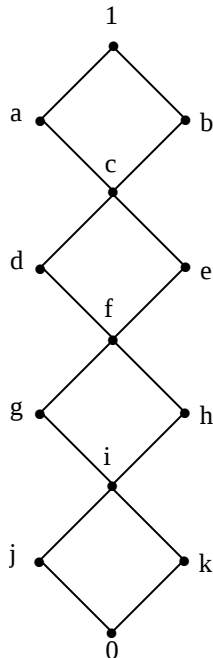
Next we have a class of distributive lattices which are Boolean algebras.

**THEOREM 6.4:** *Every Boolean algebra of order greater than or equal to four is a Smar lattice.*

**Proof:** Proof follows from the fact every Boolean algebra  $B$  is isomorphic with  $\{0, 1\} \times \{0, 1\}$  and in this product if we take  $T = \{0, 1\} \times \{0, 1\}$  then  $T$  is isomorphic with  $B$  the Boolean algebra of order four.

Hence every Boolean algebra is a Smar lattice.

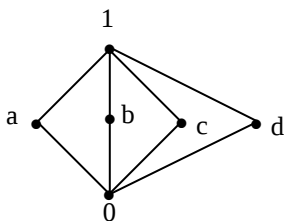
**Example 6.7:** Let  $L$  be a lattice given by the following figure.  $L$  is a Smar lattice.



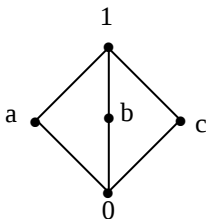
Now we proceed onto define the new notion of Smarandache sublattices and their generalization.

**DEFINITION 6.2:** Let  $L$  be a lattice. If  $L$  has a proper sublattice  $P$  and if  $P$  is a Smar lattice then we say  $P$  is a Smarandache sublattice or Smar sublattice. That is  $P$  is not the Boolean algebra of order 4 but contains a Boolean algebra of order four as a sublattice.

**Example 6.8:** Let  $L$  be lattice given by the following figure.



Take  $P =$



a sublattice of  $L$ .  $P$  is a Smar lattice. Thus  $P$  is a Smar sublattice of  $L$ .

**THEOREM 6.5:** Let  $L$  be a lattice. If  $L$  has a proper Smar sublattice  $P$  then  $L$  is a Smar lattice.

**Proof:** Given  $L$  is a lattice such that  $P \subseteq L$ ,  $P$  is a proper subset of  $L$  and  $P$  is a Smar sublattice of  $L$ . Let  $B$  be a Boolean algebra of order four contained in  $P$ , then  $B$  is also a subset of  $L$  as  $B \subseteq$

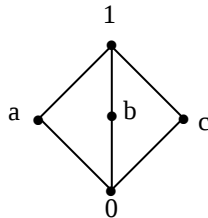
$P \subseteq L$  so  $L$  contains a proper subset which is a Boolean algebra of order 4. So  $L$  is a Smar lattice.

A natural question would be if  $L$  is a Smar lattice will  $L$  always contains a Smar sublattice. The answer is not true in general we prove this by the following theorem.

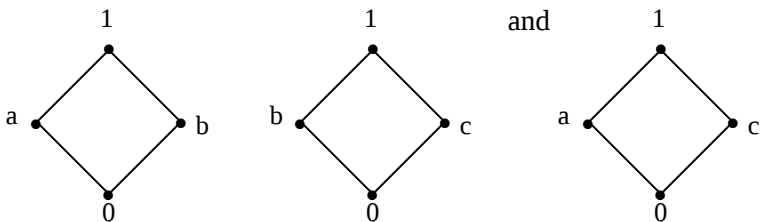
**THEOREM 6.6:** *Let  $L$  be Smar lattice.  $L$  in general need not contain a Smar sublattice.*

**Proof:** To prove the theorem we have to give a Smar lattice which has sublattices but which has no Smar sublattices. Consider the Smar lattice which is a diamond lattice.

Clearly  $L$  is Smar lattice as  $L$  contains 3 sublattices of order four all of them are isomorphic with the Boolean algebra of order 4.



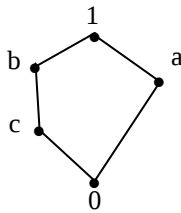
Clearly



are Boolean algebras of order four. But  $L$  has no sublattice which is a Smar sublattice as  $L$  has no sublattice of order greater than four.

So the diamond lattice is a Smar lattice which has no Smar sublattice but has sublattices.

Likewise consider the pentagon lattice P.



P is a Smar lattice as the sublattices  $\{1, a, b, 0\}$  and  $\{1, a, c, 0\}$  are such that their homomorphic image is isomorphic to the Boolean algebra of order four.

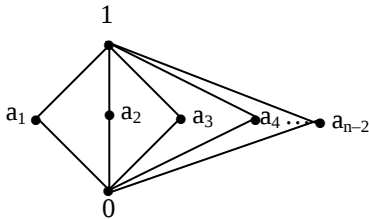
But clearly the pentagon lattice L has no sublattice which is a Smar sublattice of P.

Hence the claim.

If a pertinent to show by some examples that we can have Smar lattices which has Smar sublattices also.

**THEOREM 6.7:** *Let L be a lattice of the form  $M_n$ ,  $n > 5$ . Then L is a Smar lattice which Smar sublattices.*

**Proof:** Given  $M_n$  is a modular lattice of the form



Given  $n > 5$ . So  $M_n$  is a Smar lattice for every proper subset  $P = \{0, 1, a_i, a_{i+1}\} \subseteq M_n$ ,  $1 \leq i \leq n-2$  is a sublattice whose

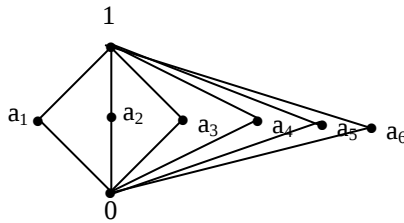
homomorphic image is isomorphic with the Boolean algebra  $B$  of order 4.

Thus  $M_n$  is a Smar lattice.

Now consider the sublattices  $P_i$  of the form  $P_i = \{0, 1, a_i, a_{i+1}, a_{i+2}\}$ . Each  $P_i$  is a Smar sublattice  $1 \leq i \leq m$ ,  $m$  a suitable number.

Thus this Smar lattice has Smar sublattices. Thus this class of lattices are Smar lattices which has Smar-sublattices.

**Example 6.9:** Let  $M_6$  be a lattice given by the following figure.

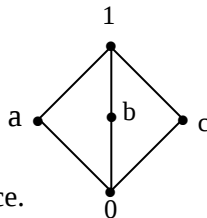


$L$  is a Smar lattice and  $L$  has Smar-sublattices.

Now we define those Smar lattices which has no Smar sublattices as simple Smarandache lattices.

**DEFINITION 6.3:** Let  $L$  be a lattice which is a Smar lattice if  $L$  has sublattices but no Smar sublattices then we call  $L$  to be a Smarandache simple lattice (Smar simple lattice) or (Simple Smar lattice).

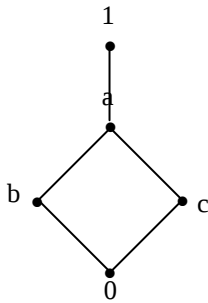
**Example 6.10:** The diamond lattice.



is a simple Smar lattice.

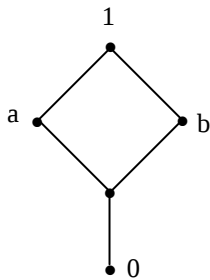


**Example 6.11:** Let  $L$  be the lattice given by the following diagram.  $L$  is a simple Smar lattice.



$L$  is a Smar-Lattice which is a simple Smar lattice.

**Example 6.12:** Let  $L =$



be a lattice given by the figure.

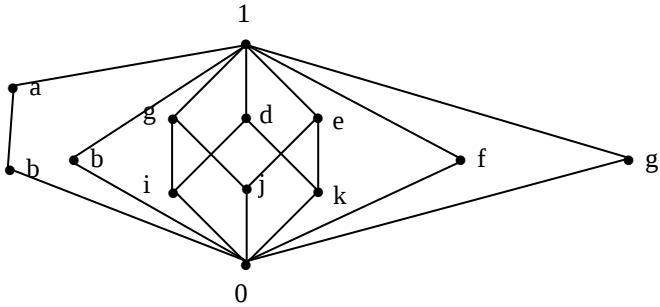
$L$  is a simple Smar lattice.

Next we proceed onto define Smar strong lattices.

**DEFINITION 6.4:** Let  $L$  be a lattice if  $L$  has a sublattice whose homomorphic image is isomorphic to a Boolean algebra of order at least eight then we call  $L$  to be a Smarandache strong lattice (S-Smar lattice).

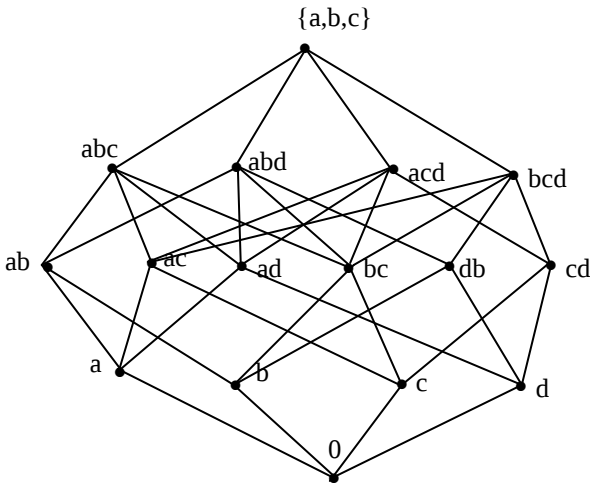
We now proceed onto give a few examples of it.

**Example 6.13:** Let  $L$  be a lattice given by the following figure.



Clearly  $L$  is a S-Smar lattice.  $L$  is not distributive.  $L$  is not modular.  $L$  is also a Smar lattice.

**Example 6.14:** Let  $L$  be a lattice given by the following figure.



We see  $L$  is a S- Smar lattice (Smar strong lattice).  
We have the following interesting.

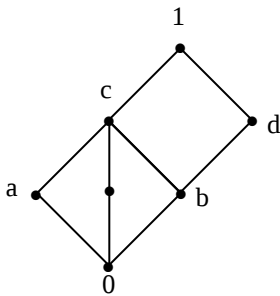
**THEOREM 6.8:** Every S-Smar lattice is a Smar lattice but a Smar-Lattice is not in general a S-Smar lattice.

**Proof:** Suppose  $L$  is a S-Smar lattice (Smar strong lattice) then clearly  $L$  has a sublattice.  $P$  whose homomorphic image is isomorphic to a Boolean algebra of order 8.

Now every Boolean algebra of order eight has a sub Boolean algebra of order four. Hence  $L$  has a sublattice of order four, whose homomorphic image is isomorphic with a Boolean algebra of order four. Thus  $L$  is a Smar lattice.

Now to show a Smar lattice in general is not a Smar strong lattice we give a example.

Consider a Smar lattice  $L$  given by the following figure.



$L$  is a Smar lattice, which is clearly not a Smar strong lattice. Since order of  $L$  is itself seven.

## Chapter Seven

# GB-ALGEBRAIC STRUCTURES

In this chapter we for the first time introduce a new algebraic structure which is not a field or a ring or a near ring or a group ring or a semigroup ring or a lattice or a Boolean algebra or a semiring or a semifield or a vector space or a linear algebra.

This is like a linear algebra having two separate algebraic structures combined in a nice mathematical way.

For constructing this algebraic structure we need a group  $G$  and a Boolean algebra  $B$ . Just as group ring are defined we define group Boolean algebra and this new algebraic structure is known as GB-algebraic structures.

**DEFINITION 7.1:** *Let  $G$  be a group and  $B = (B, +, \bullet, 0, 1)$  be a Boolean algebra. The group Boolean algebra of the group  $G$  over the Boolean algebra  $B$  consists of all finite formal sums of the form  $\sum_i b_i g_i$  ( $i$ - runs over a finite number) where  $b_i \in B$  and  $g_i \in G$  satisfies the following conditions.*

- i.  $\sum_{i=1}^n b_i g_i = \sum_{i=1}^n c_i g_i$  if and only if  $b_i = c_i$  for  $i = 1, 2, \dots, n$ ,  
 $g_i \in G$ .
- ii.  $\sum_{i=1}^n b_i g_i + \sum_{i=1}^n c_i g_i = \sum_{i=1}^n (b_i + c_i) g_i ; g_i \in G$ .
- iii.  $\left( \sum_i b_i g_i \right) \times \left( \sum_j \beta_j g_j \right) = \sum_k \gamma_k m_k$   
where  $\gamma_k = \sum_i b_i \beta_j$ ,  $g_i g_j = m_k \in G$  where  $g_i, g_j \in G$  and  
 $b_i \beta_j, \gamma_k \in B$ .
- iv.  $b_i g_i = g_i b_i$  for all  $b_i \in B$  and  $g_i \in G$ .
- v.  $b \sum_{i=1}^n b_i g_i = \sum (bb_i) g_i$  for  $bb_i \in B$  and  $\sum b_i g_i \in BG$ .

$BG$  is a new special algebraic structure defined as the  $GB$  algebraic structure.  $0 \in B$  acts as the additive identity of  $GB$ . Since  $1 \in B$  we have  $G = 1.G \subseteq BG$  and  $B.e = B \subseteq BG$  where  $e$  is in identity element of  $G$ .

We denote  $1.e = 1.1 = 1$  which is the multiplicative identity of  $BG$ .

**Remark 7.1:** We see  $BG$  is not any of the known algebraic structures. It is not a Boolean algebra as  $G \subseteq BG$ .  $BG$  is not a group as  $BG$  is not a ring as we do not have inverse elements under addition. So  $BG$  is the special  $GB$ -algebraic structure.

We give some examples before we proceed on to define more properties about the  $GB$ -algebraic structures.

**Example 7.1:** Let  $B = \{0, 1\}$  be the Boolean algebra of order two.  $G = \langle g \mid g^3 = 1 \rangle$  be the cyclic group of order three.  $BG$  be the  $GB$ -algebraic structure.

$$BG = \{0, 1, g, g^2, 1+g, 1+g^2, \dots, 1+g+g^2\}.$$

We see  $(g + g^2) + (g + g^2) = g + g^2$  as  $1 + 1 = 1$  in  $B$ .

$$\begin{aligned} (1 + g + g^2) + (g + g^2) &= 1 + (g + g^2) + (g + g^2) \\ &= 1 + g + g^2. \end{aligned}$$

$$\begin{aligned} g + g^2 + g &= g^2 + g \\ (\text{as } g + g &= g) \\ (1 + g + g^2)(1 + g) &= 1 + g + g^2 + g + g^2 + 1 \\ &= 1 + g + g^2 \\ g^2(1 + g) &= g^2 + 1 \text{ and so on.} \end{aligned}$$

We see  $BG$  has eight elements and every element  $x$  in  $BG$  is such that  $x + x = x$ .

But every element  $x$  in  $BG$  need not in general be such that  $x.x = x$

$$\begin{aligned} \text{For take } x = g + g^2; \quad x.x &= (g + g^2)(g + g^2) \\ &= g^2 + 1 + 1 + g \\ &= 1 + g + g^2 \neq g + g^2. \end{aligned}$$

Hence the claim.

Thus  $BG$  is not a Boolean algebra.

Further as  $(1 + g + g^2)(1 + g + g^2) = 1 + g + g^2$  we see  $BG$  is not a group.

Also  $0 \in BG$  so  $BG$  is not a group.

**Example 7.2:** Let  $B = \{0, 1\}$  be a Boolean algebra of order two and  $G = \langle g \mid g^2 = 1 \rangle$ . Then  $BG = \{0, 1, g, 1 + g\}$  be the group Boolean algebra. Here  $(1 + g)(1 + g) = 1 + g$  so  $BG$  is not a group. As  $g.g = 1$ ,  $BG$  is not a Boolean algebra. But  $B \subseteq BG$  and  $G \subseteq BG$ .

**Example 7.3:** Let  $B = \{0, 1\}$  be the Boolean algebra of order two and  $G = S_3$  be the symmetric group of degree three. Then  $BG$  is a GB-algebraic structure where  $BG = \{0, 1, p_1, p_2, p_3, p_4, p_5, 1+p_1, \dots, 1+p_5, \dots, p_4 + p_5, \dots, 1 + p_1 + p_2 + p_3 + p_4 + p_5\}$ . Clearly  $BG$  has  $2^6$  elements. We see there exists  $x, y \in BG$  such that  $xy \neq yx$ . For  $p_4, p_2 \neq p_2, p_4$  where  $p_2, p_4 \in BG$ .

In view of this we make the following definition.

**DEFINITION 7.2:** Let  $BG$  be a GB-structure where  $B$  is a Boolean algebra and  $G$  is a group. If in  $BG$  for every  $x, y \in BG$  we have  $xy = yx$  then we call the GB-algebraic structure to be commutative. If  $BG$  has at least one pair of elements  $x, y$  such that  $xy \neq yx$  then we say the GB-algebraic structure is non commutative.

The examples 7.1 and 7.2 are commutative GB-algebraic structures where as example 7.3 is a non commutative GB-algebraic structure.

We give the following interesting theorem.

**THEOREM 7.1:** Let  $B$  be any Boolean algebra.  $BG$  the GB-algebraic structure is commutative if and only if  $G$  is a commutative group.

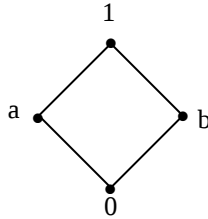
**Proof:** Given  $BG$  is the GB-algebraic structure so  $B \subseteq BG$  and  $G \subseteq BG$  where  $B$  is the Boolean algebra and  $G$  is a group. Suppose  $BG$  is commutative then we know  $G \subseteq BG$  so  $G$  is a commutative group.

Now if  $G$  is a commutative group then we see from the very definition of the GB-algebraic structure.  $BG$  is a commutative algebraic structure.

**Corollary 7.1:**  $BG$  is a non commutative algebraic structure if and only if  $G$  is a non commutative group.

**Proof:** Similar to the theorem 7.1.

**Example 7.4:** Let  $B = \{0, a, b, 1\}$  be the Boolean algebra of four elements.



$$a+b = 1$$

$$a.b = 0$$

$$a.a = a$$

$$b.b = b.$$

Let  $G = \langle g \mid g^2 = 1 \rangle$  be the cyclic group of order two.  $BG = \{0, 1, a, b, g, ag, bg, 1+g, a+g, b+g, 1+ag, 1+bg, b+ag, bg+a, a+ag, b+bg\}$ .

The GB-algebraic structure is commutative and has 16 elements.

$b.ag = 0$  and  $(a+ag)(b+bg) = 0$ . Thus we see the algebraic structure has zero divisors.

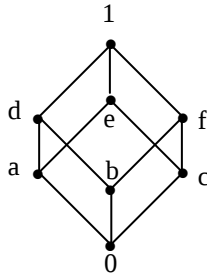
$$\begin{aligned}
 \text{Also } (a+bg)(ag+b) &= a.ag + bg.ag + a.b + b.bg \\
 &= ag + 0 + 0 + bg \\
 &= (a+b)g = 1.g = g.
 \end{aligned}$$

Thus it is interesting to see that the group element  $g$  in  $G$  is got as a product of two distinct elements from  $BG \setminus G$ .

**DEFINITION 7.3:** Let  $BG$  be the group Boolean algebra (GB-algebraic structure) of the group  $G$  over the Boolean algebra  $B$ . We say  $x \neq 0$  is a zero divisor in  $BG$  if there exists a  $y \in BG \setminus \{0\}$  such that  $x.y = 0$ .

**Example 7.5:** Let  $B = \{0, a, b, c, d, e, f, 1\}$  be the Boolean algebra of order 8 given by the following figure.





Let  $G = \langle g \mid g^2 = 1 \rangle$  be the cyclic group of order two.  $BG = \{0, 1, a, b, c, d, e, f, g, ag, bg, cg, dg, eg, fg, \dots, 1+g, 1+ag, 1+bg, \dots, 1+fg, a+g, b+g, \dots, f+g, a+ag, b+bg, \dots, f+fg, a+bg, a+cg, \dots, a+fg, b+ag, b+cg, \dots, b+fg, c+ag, c+bg, \dots, c+fg, d+ag, d+bg, \dots, d+fg, e+ag, e+bg, \dots, e+fg, f+ag, f+bg, \dots, f+fg\}$ ,  $o(BG) = 64 = 8^2$ .

We see  $BG$  has zero divisors and idempotents.

We prove the following interesting theorem.

**THEOREM 7.2:** *Let  $G$  be any finite group.  $B = \{0, 1\}$  be the Boolean algebra of order two.  $BG$  has no zero divisors but has non trivial idempotents.*

**Proof:** Since  $B = \{0, 1\}$  has no elements  $x, y$  such that  $x.y = 0$ ,  $x, y \in B \setminus \{0\}$  we see in  $BG$  no element  $x \neq 0$  in  $BG$  has a  $y$  in  $BG \setminus \{0\}$  such that  $x.y = 0$ .

Further  $1 + g + g^2 + \dots + g^{t-1} \in BG$  where  $t \mid o(G)$  and is such that  $g^t = 1$  is an idempotent in  $BG$  for  $(1 + g + \dots + g^{t-1})(1 + g + \dots + g^{t-1}) = 1 + g + \dots + g^{t-1}$  as  $1 + 1 = 1$  in  $B$ .

Hence the claim.

**Example 7.6:** Let  $B = \{0, 1\}$  be the Boolean algebra of order two and  $G = \langle g \mid g^9 = 1 \rangle$  be cyclic group of order nine. Now  $BG$  be the GB-algebraic structure. Take  $x = 1 + g + \dots + g^8$  in  $BG$   $x^2 = (1 + g + \dots + g^8)^2 = (1 + g + \dots + g^8)$ .

Further  $x.y \neq 0$  for any  $x, y \in BG \setminus \{0\}$

$$\begin{aligned} \text{Also that } (1 + x^3 + x^6)(1 + x^3 + x^6) \\ = 1 + x^3 + x^6 + x^3 + x^6 + 1 + x^6 + 1 + x^3 \\ = 1 + x^3 + x^6. \end{aligned}$$

Hence the claim.

$$\begin{aligned} (1 + x + x^2)(1 + x + x^2) \\ = 1 + x + x^2 + x + x^2 + x^3 + x^2 + x^3 + x^4 \\ = 1 + x + x^2 + x^3 + x^4. \end{aligned}$$

**THEOREM 7.3:** Let  $G$  be a group of finite order.  $B = \{0, 1\}$  be the Boolean algebra of order two.  $BG$  has non trivial idempotents.

**Proof:** Let  $o(G) = n$  and  $G = \{1, g_1, g_2, \dots, g_{n-1}\}$  where each  $g_i$  is distinct (i.e.,  $g_i = g_j$  and only if  $i = j$ ,  $1 \leq i, j \leq n-1$ ). Take  $x = 1 + g_1 + \dots + g_{n-1}$  in  $BG$ . We see  $x^2 = (1 + g_1 + \dots + g_{n-1})^2 = 1 + g_1 + \dots + g_{n-1}$  i.e.,  $x$  is an idempotent in  $BG$  hence the claim.

**Example 7.7:** Let  $G = S_3$  be the symmetric group of degree three group and  $B = \{0, 1\}$  be the Boolean algebra of order 2.  $BG$  be the GB-algebraic structure.

$x = 1 + p_1 + \dots + p_5$  in  $BG$  is such that  $x^2 = x$ .

**Example 7.8:** Let  $G = D_{2,6} = \{a.b \mid a^2 = b^6 = 1, bab = a\} = \{1, a, b, b^2, \dots, b^5, ab, ab^2, ab^3, ab^4, ab^5\}$  be the dihedral group of order six.  $B = \{0, 1\}$  be the Boolean algebra of order two.  $BG$  be the GB-algebraic structure.

Take  $x = 1 + b + b^2 + \dots + b^5 \in BG$ ;  $x^2 = x$ ;  $y = 1 + ab^2 \in BG$  is such that  $y^2 = (1 + ab^2)^2$

$$\begin{aligned} &= 1 + ab^2 + ab^2 + ab^2ab^2 \\ &= 1 + ab^2 + ab^2 + 1 \\ &= 1 + ab^2. \end{aligned}$$

BG has atleast 8 idempotents. However it can be easily verified that BG has no zero divisors.

**Example 7.9:** Let  $G = S_3$  be the symmetric group of degree three i.e.,  $G = S_3 = \{1, p_1, p_2, \dots, p_5\}$  and  $B = \{0, a, b, 1\}$  be the Boolean algebra of order four. BG the GB-algebraic structure. BG has both non trivial zero divisors and nontrivial idempotents.

$$\begin{aligned} & \text{Take } (ap_1 + ap_2) (bp_1 + bp_2) \\ &= ab p_1^2 + abp_1p_2 + ab p_2^2 + abp_2p_1 \\ &= 0 + 0 + 0 + 0. \end{aligned}$$

Thus  $ap_1 + ap_2, bp_1 + bp_2 \in BG$  is a zero divisor in BG.

$$\begin{aligned} & \text{Also } (1 + p_4 + p_5) (1 + p_4 + p_5) \\ &= 1 + p_4 + p_5 \text{ in BG.} \end{aligned}$$

$$\begin{aligned} & (a + bp_1 + cp_2) (b + cp_1 + ap_2) \\ &= 0 + bp_1 + 0.p_2 + 0.p_1 + 0.p_1 + cp_2 p_1 + ap_2 + 0 p_1p_2 + 0 p_2^2 \\ &= bp_1 + ap_2 + cp_4 \in BG. \end{aligned}$$

We see every product in BG does not in general lead to zero divisor in BG.

Now we proceed onto define GB-sub algebraic structures.

**DEFINITION 7.4:** Let  $G$  be a group and  $B$  be a Boolean algebra. We say BG the group Boolean algebra has a GB-sub algebraic structure  $S$  if  $S$  contains a proper subset  $H$  such that  $H$  is a subgroup of  $G$  and a proper subset  $T$  in  $S$  such that  $T = B$  or  $T$  is a subBoolean algebra of  $B$  or if  $S$  contains  $G$  as a proper subset but  $B \subseteq S$  only a proper subset  $T$  of  $S$  which is a sub Boolean algebra of  $B$ ; i.e.,  $T \subseteq B$  and  $T \neq B$ .

We illustrate this by the following examples.

**Example 7.10:** Let  $B = \{0, 1\}$  be the Boolean algebra of order two.  $G = D_{2,6}$  be the dihedral group of order 12. BG be the GB-

algebraic structure. Take  $S = \{0, 1, b, b^2, b^3, b^4, b^5, 1+b, 1+b^2, 1+b^3, 1+b^4, 1+b^5, b+b^2, b+b^3, b+b^4, b+b^5, b^2+b^3, b^2+b^4, b^2+b^5, b^3+b^4, b^4+b^5, b^3+b^5, 1+b+b^2, \dots, 1+b+b^2+b^3+b^4+b^5\} \subseteq BG$  is a sub GB-algebraic structure as  $B = \{0, 1\} \subseteq S$  and  $H = \{1, b, b^2, \dots, b^5\} \subseteq S$ . Infact BG has several GB-sub algebraic structures, like  $T_1 = \{0, 1, ab, 1+ab\}$ ,  $T_2 = \{0, 1, a, 1+a\}$  and  $T_i = \{1, ab_i, 0, 1+ab_i; 1 \leq i \leq 5\}; i = 1, 2, \dots, 5$ .

**THEOREM 7.4:** *Let  $G = \langle g \mid g^p = 1 \rangle$  be a cyclic group of prime order and  $B = \{0, 1\}$  be the Boolean algebra of order two. Then BG the GB-algebraic structure has no proper GB algebraic sub structures.*

**Proof:** Given G is a cyclic group of prime power order, so G has no proper subgroup. Further  $B = \{0, 1\}$  is a Boolean algebra of order two so B has no proper subalgebra.

So the GB-algebraic structure BG has no proper GB-sub algebraic structures. Thus we have a large class of GB-algebraic structures which has no GB-algebraic sub structures, given by BG where  $B = \{0, 1\}$  and G a cyclic group of prime order.

We illustrate this situation by the following examples.

**Example 7.11:** Let  $G = \langle g \mid g^7 = 1 \rangle$  be a cyclic group of order 7 and  $B = \{0, 1\}$  be a Boolean algebra of order two.  $BG = \{0, 1, g, \dots, g^6, 1+g, 1+g^2, \dots, 1+g+g^2 + g^3 + g^4 + g^5 + g^6\}$  be the group Boolean algebra of B over G. BG has no proper GB-algebraic sub structure.

Another natural question would be can we characterize BG-algebraic structures which has proper sub GB-algebraic structure. The answer is yes which is given in the form of the following theorem.

**THEOREM 7.5:** *Let G be a group having proper subgroups and B be any Boolean algebra. Then BG the GB-algebraic structure has proper GB-algebraic substructure.*

**Proof:** Given  $G$  has proper subgroups.  $BG$  be the GB-algebraic structure. Let  $H \neq \{e\}$  be proper subgroup of  $G$ .  $BH$  is a GB-algebraic structure and clearly  $BH \subset BG$ .

Hence the claim.

**THEOREM 7.6:** *Let  $G$  be any group.  $B$  a Boolean algebra of order greater than or equal to four. Then  $BG$  the GB-algebraic structure has proper GB-algebraic substructures.*

**Proof:** Given  $G$  is any group and  $B$  a Boolean algebra of order greater than or equal to four, so  $B$  contains  $T = \{0, 1\}$  as a proper Boolean subalgebra.

Now  $TG$  is a GB-algebraic substructure which is properly contained in  $BG$ ; hence the claim.

**Example 7.12:** Let  $B = \{0, a, b, 1\}$  be a Boolean algebra,  $G = A_4$ , the alternating subgroup of the symmetric group  $S_4$ .  $BG$  is the GB-algebraic structure. Now take  $T = \{0, 1\}$ ,  $TA_4$  is a GB-algebraic structure contained in  $BG$ . So  $BG$  has GB-algebraic substructure.

**Example 7.13:** Let  $G = \langle g \mid g^{17} = 1 \rangle$  be the group of order 17.  $B = \{0, a, b, 1\}$  be the Boolean algebra of order four.  $BG$  be the GB-algebraic structure. Take  $T = \{0, 1\}$ ,  $TB$  is a GB-algebraic structure such that  $TB \subseteq GB$  so  $BG$  has GB-algebraic substructures.

Now we define a new notion.

**DEFINITION 7.5:** *Let  $B$  be a Boolean algebra and  $G$  be any group.  $BG$  be the GB-algebraic structure. If  $BG$  has no GB-algebraic substructure then we call  $BG$  to be a simple GB-algebraic structure.*

We have a class of simple GB algebraic structures. Take  $B = \{0, 1\}$  to be the Boolean algebra of order two and

$G = \langle g \mid g^p = 1 \rangle$ ;  $p$  a prime;  $BG$  the group Boolean algebra is a simple GB-algebraic structure.

We also have a class of GB-algebraic structure which are not simple GB-algebraic structure. Take  $G$  be a group of any finite order  $n$ ,  $n$  a composite number.  $B$  be any Boolean algebra then  $BG$  the GB-algebraic structure is not a simple GB-algebraic structure.

Now we give some more illustrations.

**Example 7.14:** Let  $G = \langle g \mid g^{31} = 1 \rangle$  and  $B = \{0, 1\}$  be the group of order 31 and the Boolean algebra of order two respectively. Then  $BG$  the GB-algebraic structure is simple. Further  $BG$  has no zero divisors.  $o(BG) = 2^{31}$ .

Also  $(1 + g + \dots + g^{31})^2 = (1 + g + \dots + g^{31})$  i.e.,  $BG$  has no trivial idempotents. Now take  $(1+g) \cdot (g^2 + g^4) = g^2 + g^4 + g^3 + g^5$  where  $1 + g, g^2 + g^4 \in BG$ .

Take  $g + g^2$  and  $g^2 + g^7$  in  $BG$ . Consider  $(1 + g^2)(g^2 + g^7) = g^2 + g^4 + g^9 + g^7$ .

Suppose  $1 + g + g^2 + g^3 + g^7, g^2 + g^5 + g + g^{11} \in BG$ .

Consider their product

$$(1 + g + g^2 + g^3 + g^7)(g^2 + g^5 + g + g^{11}) = g^2 + g^3 + g^4 + g^5 + g^9 + g^5 + g^6 + g^7 + g^8 + g^{12} + g + g^2 + g^3 + g^4 + g^8 + g^{11} + g^{12} + g^{13} + g^{14} + g^{18} = g + g^2 + g^3 + g^4 + g^5 + g^6 + g^7 + g^8 + g^9 + g^{11} + g^{12} + g^{13} + g^{14} + g^{18}.$$

Now we define one more new concept.

**DEFINITION 7.6:** Let  $BG = \left\{ \sum_i b_i g_i \mid b_i \in B \text{ and } g_i \in G \right\}$  be the GB-algebraic structure of the group  $G$  over the Boolean algebra  $B$ .

Let  $\alpha = \sum b_g g \in BG$  we define content of  $\alpha$  denoted by  $\text{cont}\alpha = \{g \in G \mid b_g \neq 0\}$ . Clearly  $\text{cont}\alpha$  is a finite subset of  $G$ .

If we have  $\alpha$  in  $BG$  to be central in  $BG$  and if  $x \in \text{cont}\alpha$ . If  $y \in G$  then  $x^y = y^{-1}xy \in \text{cont}y^{-1}xy \in \text{cont}y^{-1}\alpha y = \text{cont}\alpha$ .

Since  $\text{cont}\alpha$  is finite we have only a finite number of distinct  $x^y$  with  $y \in G$ .

It is interesting to study the set of elements  $x \in G$  with this property.

We see for  $\alpha, \beta \in BG$  where  $\alpha = \sum b_g g$  and  $\beta = \sum c_h h$ ,  $h, g \in G$  and  $b_g, c_h \in B$  the cardinality of  $\text{cont}\alpha$  and  $\text{cont}\beta$  is such that  $|\text{cont}\alpha\beta| \leq |\text{cont}\alpha| |\text{cont}\beta|$ .

We first illustrate this by some simple examples.

**Example 7.15:** Let  $G = S_3 = \{1, p_1, p_2, p_3, p_4, p_5\}$  be the symmetric group of degree three and  $B = \{0, 1\}$  be the Boolean algebra of order two.  $BG$  the GB-algebraic structure.

Let  $\alpha = \{p_1 + p_2 + p_3\}$  and  $\beta = 1 + p_4$  be in  $BG$ .

We see content of  $\alpha = \{p_1, p_2, p_3\}$  and content of  $\beta = \{1, p_4\}$ .

$$\begin{aligned} \text{Now } \alpha\beta &= (p_1 + p_2 + p_3)(1 + p_4) \\ &= p_1 + p_2 + p_3 + p_3 + p_1 + p_2 \\ &= p_1 + p_2 + p_3. \end{aligned}$$

Thus content of  $\alpha\beta = \{p_1, p_2, p_3\}$  and  $|\text{cont}\alpha\beta| = 3$  but  $|\text{cont}\alpha| |\text{cont}\beta| = 3 \cdot 2 = 6$ .

Hence we see  $|\text{cont}\alpha\beta| \leq |\text{cont}\alpha| |\text{cont}\beta|$ .

**Example 7.16:** Let  $G = \langle g \mid g^{10} = 1 \rangle$  be the cyclic group of order 10 and  $B = \{0, 1\}$  be the Boolean algebra of order two BG be the GB-algebraic structure.

$$BG = \{0, 1, g, g^2, \dots, g^9, 1+g, \dots, 1+g^9, g+g^2, \dots, g^8+g^9, \dots, 1+g + \dots g^9\}.$$

Take  $\alpha = \{1 + g + g^2 + g^3 + g^4\}$  and  $\beta = \{g + g^5 + g^6 + g^7 + g^8\}$  in BG.

$$\text{Now cont } \alpha = \{1, g, g^2, g^3, g^4\}$$

$$|\text{cont } \alpha| = 5. \text{ cont } \beta = \{g, g^5, g^6, g^7, g^8\}; |\text{cont } \beta| = 5$$

$$\begin{aligned} \alpha\beta &= (1 + g + g^2 + g^3 + g^4)(g + g^5 + g^6 + g^7 + g^8) \\ &= g + g^2 + g^3 + g^4 + g^5 + g^6 + g^7 + g^8 + g^9 + g^6 + g^7 + g^8 + g^9 + 1 + g^7 + g^8 + g^9 + 1 + g + g^8 + g^9 \\ &\quad + 1 + g + g^2 \\ &= 1 + g + g^2 + g^3 + g^4 + g^5 + g^6 + g^7 + g^8 + g^9. \end{aligned}$$

$$\text{Cont } \alpha\beta = \{1, g, g^2, \dots, g^9\} \text{ and } |\text{cont } \alpha\beta| = 10.$$

$$\text{Thus } |\text{cont } \alpha\beta| \leq |\text{cont } \alpha| |\text{cont } \beta|.$$

**Example 7.17:** Let  $G = \{g \mid g^6 = 1\}$  be the cyclic group of order 6 and  $B = \{0, a, b, 1\}$  be the Boolean algebra of order four.  $BG = \{0, a, b, 1, g, g^2, \dots, g^5, ag, ag^2, \dots, ag^5, bg, bg^2, \dots, bg^5, \dots, 1 + g + g^2 + g^3 + g^4 + g^5, a + ag + ag^2 + ag^3 + ag^4 + ag^5, b + bg + bg^3 + bg^4 + ag^5\}$  be the GB-algebraic structure.

Take  $\alpha = a + ag + \dots + ag^5$  and  $\beta = b + bg + \dots + bg^5$  in BG.

$$\text{Cont } \alpha = \{1, g, g^2, g^3, g^4, g^5\} = G \text{ and } |\text{cont } \alpha| = |G| = 6.$$

$$\text{Cont } \beta = \{1, g, g^2, \dots, g^5\} = G \text{ and } |\text{cont } \beta| = |G| = 6.$$

We find  $\alpha\beta$ .



$$\alpha\beta = (a + ag + \dots + ag^5) (b + bg + \dots + bg^5) = a (1 + g + \dots + g^5) b (1 + g + \dots + g^5) = ab (1 + g + \dots + g^5)^2 = ab (1 + g + g^2 + \dots + g^5) = 0 \text{ as } a.b = 0 \text{ in } B.$$

Thus  $\alpha\beta = \phi$ ,  $|\text{cont } \alpha\beta| = 0$ .

We see  $|\text{cont } \alpha\beta| \leq |\text{cont } \alpha| |\text{cont } \beta|$ .

Let  $\alpha = (1 + g^2 + g^4)$  and  $\beta = (b + bg^2 + bg^4)$  in BG we see

$\text{Cont } \alpha = \{1, g^2, g^4\}$  and  $|\text{cont } \alpha| = 3$ .

$\text{Cont } \beta = \{1, g^2, g^4\}$  and  $|\text{cont } \beta| = 3$ .

$$\begin{aligned} \alpha\beta &= (1 + g^2 + g^4) (b^1 + bg^2 + bg^4) \\ &= b + bg^2 + bg^4 + bg^2 + bg^4 + bg^4 + bg^6 + bg^6 \quad (g^6 = 1) \\ &= b + bg^2 + bg^4 \end{aligned}$$

$\text{Cont } \alpha\beta = \{1, g^2, g^4\}$  and  $|\text{cont } \alpha\beta| = 3$

Thus  $|\text{cont } \alpha\beta| \leq |\text{cont } \alpha| |\text{cont } \beta|$ .

We see  $\alpha\beta = \beta$  in this case. We cannot say  $\alpha\beta = \beta$  can be simplified as  $\alpha\beta - \beta = 0$  or  $(\alpha-1)\beta = 0$  as in GB-algebraic structure the notion of  $-\alpha$  for any  $\alpha \in BG$  does not have any meaning as B is a Boolean algebra.

Now before we prove  $|\text{cont } \alpha\beta| \leq |\text{cont } \alpha| |\text{cont } \beta|$  we first prove in case of finite groups G we have  $|\text{cont } \alpha| |\text{cont } \beta| \leq |G| |G|$  and  $|\text{cont } \alpha\beta|$  can be 0.

Thus we have  $0 \leq |\text{cont } \alpha\beta| \leq |\text{cont } \alpha| |\text{cont } \beta| \leq |G| |G|$ .

**THEOREM 7.7:** *Let G be any finite group.  $B = \{0, 1\}$  be the Boolean algebra of order two. BG the GB-algebraic structure. Let  $\alpha, \beta \in BG$  be such that  $\text{cont } \alpha \neq \phi$  and  $\text{cont } \beta \neq \phi$  then  $\text{cont } \alpha\beta \neq \phi$ ,*

i.e., if  $|\text{cont } \alpha| \neq \phi$  and  $|\text{cont } \beta| \neq \phi$  then  $|\text{cont } \alpha\beta| \neq \phi$ .

**Proof :** Given  $B = \{0, 1\}$  is a Boolean algebra of order two and  $G$  a finite group with  $\alpha, \beta \in BG$  such that  $|\text{cont } \alpha| \neq \phi$  and  $|\text{cont } \beta| \neq \phi$ . Now in  $\text{cont } \alpha\beta$  we see no term can vanish as 0, as  $B$  has no zero divisor coefficient terms in  $\alpha$  and  $\beta$  (i.e.,  $\text{cont } \alpha$  and  $\text{cont } \beta$  remain as non zero). So  $\text{cont } \alpha\beta \neq \{0\}$ .

Hence  $|\text{cont } \alpha\beta| \neq \phi$  in case  $B$  is a boolean algebra of order two.

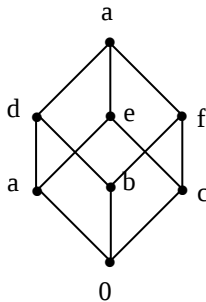
Now we cannot assert in this way in case of Boolean algebras of order greater than two for these Boolean algebras  $B$  always have complements of elements in  $B$  such that their product is always zero.

We first illustrate this by an example before we prove any result in this direction.

**Example 7.18:** Let  $G = \langle g \mid g^8 = 1 \rangle$  be the cyclic group of order eight and  $B = \{0, a, b, c, d, e, f, 1\}$  be a Boolean algebra of order eight.

Let  $BG$  be the GB-algebraic structure. We can have  $\alpha, \beta \in BG$  with  $|\text{cont } \alpha| \neq 0$  and  $|\text{cont } \beta| \neq 0$  but  $|\text{cont } \alpha\beta| = 0$ .

For take  $B$  to be the Boolean algebra given by the figure.



We have  $a.b = 0 = b.c = a.c = d.c = e.b = f.a$ .

Let  $\alpha = ag + a$  and  $\beta = bg^2 + bg + bg^3 + fg^5 + f$  be in  $BG$ .

Clearly  $\text{cont } \alpha = \{1, g\}$ ,  $|\text{cont } \alpha| = 2$ ,  $\text{cont } \beta = \{1, g, g^2, g^3, g^5\}$  and  $|\text{cont } \beta| = 5$ .

Now  $\alpha\beta = (ag + a)(bg^2 + bg + bg^3 + fg^5 + f) = abg^4 + abg^2 + bag^2 + abg + abg^4 + abg^3 + afg^6 + afg^5 + afg + af = 0$  as  $ab = fa = 0$ .

$\text{Cont } \alpha\beta = \phi$  so  $|\text{cont } \alpha\beta| = 0$ .

Thus in the case of Boolean algebras of order greater than two we can have  $|\text{cont } \alpha\beta| = 0$  without  $|\text{cont } \alpha| = 0$  and  $|\text{cont } \beta| = 0$ .

We see some more interesting properties about  $\text{cont } \alpha$ ,  $\alpha \in \text{BG}$ .

Let  $\alpha, \beta \in G$  where  $\text{BG}$  is a GB-algebraic structure  $|\text{cont } \alpha| + |\text{cont } \beta| \geq |\text{cont } (\alpha + \beta)|$ .

We first illustrate this by the following examples.

**Example 7.19:** Let  $G$  be the finite dihedral group  $D_{2.6}$  and  $B = \{0, 1\}$  be the Boolean algebra of order two.  $\text{BG}$  be the GB-algebraic structure.

$D_{2.6} = \{1, a, b, b^2, \dots, b^5, ab, ab^2, \dots, ab^5\}$  be the given Dihedral group.

$\text{BG} = \{\sum \alpha_i g_i; \alpha_i \in B \text{ and } g_i \in D_{2.6}\}$  be the GB-algebraic structure.

Let  $\alpha = a + b + ab^3 + ab^5 + b^2$  and  $\beta = a + ab^2 + ab^3 + b^2 + b^5 + b^3$  be elements of  $\text{BG}$ .

Now  $\text{cont } \alpha = \{a, b, ab^3, b^2, ab^5\}$  so  $|\text{cont } \alpha| = 5$  and  $\text{cont } \beta = \{a, ab^2, ab^3, b^2, b^5, b^3, b\}$  so that  $|\text{cont } \beta| = 7$ .

$$\begin{aligned}\text{Now } \alpha + \beta &= (a + b + ab^3 + ab^5 + b^2 + a + ab^2 + ab^3 + b^2 + \\ &\quad b^5 + b^3 + b \\ &= a + b + ab^2 + ab^3 + b^2 + b^3 + ab^5.\end{aligned}$$

$$\begin{aligned}\text{cont } (\alpha + \beta) &= \{a, b, ab^2, ab^3, b^2, b^3, b^5, ab^5\} \text{ and} \\ |\text{cont } (\alpha + \beta)| &= 8.\end{aligned}$$

$$|\text{cont } (\alpha + \beta)| \leq |\text{cont } \alpha| + |\text{cont } \beta|.$$

**Example 7.20:** Let  $G = \langle g \mid g^{11} = 1 \rangle$  be the group of order 11 and  $B = \{0, a, b, 1\}$  be the Boolean algebra of order four.  $BG$  be the GB-algebraic structure.

Let  $\alpha = (1 + g + g^{10} + g^2 + bg^3 + g^4 + ag^5 + bg^6 + g^7 + g^8 + bg^9)$  and

$\beta = (a + bg + g^2 + ag^3 + bg^4 + g^5 + ag^6 + bg^7 + g^8 + bg^9 + ag^{10})$  be take from  $BG$ . Now  $\text{cont } \alpha = \{1, g, g^{10}, g^2, g^3, g^4, g^5, g^6, g^7, g^8, g^9\}$   $\text{cont } \beta = \{1, g, g^2, g^3, g^4, \dots, g^{10}\}$ .

We see  $|\text{cont } \alpha| = |\text{cont } \beta| = 11 = |G|$ .

Now  $\alpha + \beta = (1 + g + g^2 + bg^3 + g^4 + ag^5 + bg^6 + g^7 + g^8 + bg^9 + ag^{10})$

$$\begin{aligned}&(a + bg + g^2 + ag^3 + bg^4 + g^5 + ag^6 + bg^7 + g^8 + bg^9 + ag^{10}) = \\ &(1+a) + (a+b)g + (1+1)g^2 + (b+a)g^3 + (1+b)g^4 + (a+1)g^5 + \\ &(b+a)g^6 + (1+b)g^7 + (a+1)g^8 + (b+b)g^9 + (a+a)g^{10}\end{aligned}$$

$$= 1 + g + g^2 + g^3 + g^4 + g^5 + g^6 + g^7 + g^8 + bg^9 + ag^{10}.$$

$$\text{Cont } \alpha + \beta = \{1, g, g^2, g^3, \dots, g^{10}\}$$

Thus  $|\text{cont } \alpha + \beta| < |\text{cont } \alpha| + |\text{cont } \beta|$ .

Now it is interesting to see  $|\text{cont } \alpha + \beta| \neq 0$  if  $|\text{cont } \alpha| \neq 0$  (or)  $|\text{cont } \beta| \neq 0$ ; for in a Boolean algebra  $B$  if  $a, b \in B \setminus \{0\}$ ;  $a+b \neq 0$ .

We further see  $|\text{cont } (\alpha+\beta)|$  is always less than or equal to order of  $G$ .

Thus  $o(G) \leq |\text{cont } (\alpha+\beta)| \leq |\text{cont } \alpha| + |\text{cont } \beta| \leq 2 o(G) = 2|G|$ .

The proof of the above inequality is left as an exercise for the reader.

Now we proceed onto define the notion of invariant and universally invariant elements of the GB-algebraic structure.

**DEFINITION 7.7:** *Let  $G$  be any group of  $B$  be any Boolean BG the GB-algebraic structure.  $\alpha \in BG$  is said to be invariant in BG if there exists a  $\beta \in BG \setminus \{0\}$  such that  $\alpha.\beta = \alpha$ .*

If  $\alpha \in BG$  is such that  $\alpha.\beta = \alpha$  for all  $\beta \in BG \setminus \{0\}$  then we say  $\alpha$  is an universally invariant element of BG.

It is interesting to see  $0 \in BG$  is an universally invariant element of BG.

**THEOREM 7.8:** *Let BG be a GB-algebraic structure where  $B = \{0, 1\}$  is a Boolean algebra of order two and  $G$  any group.*

*If  $\alpha \in BG$  is universally invariant element of BG, then  $\alpha$  is an invariant element of BG. Further an invariant element of BG in general is not an universally invariant element of BG.*

**Proof:** From the very definition of the universally invariant and invariant element of BG we see every universally invariant element of BG is an invariant element of BG.

To prove that converse we give an example.

Take  $G = D_{2,9} = \{1, a, b \mid a^2 = b^9 = 1, bab = a\}$  to be the dihedral group of order 18 and  $B = \{0, 1\}$  the Boolean algebra of order two. BG be the GB-algebraic structure.

Take  $\alpha = 1 + b + b^2 + \dots + b^8$  in BG. Now there is a  $\beta = 1 + b^3 + b^6 \in BG$  such that  $\alpha\beta = \alpha$ .

Take  $\gamma = a + b$  in BG.

$$\begin{aligned} \text{Now } \alpha\gamma &= (1 + b + b^2 + b^3 + \dots + b^8)(a + b) \\ &= a + ba + b^2a + b^3a + \dots + b^8a + b + b^2 + b^3 + b^4 + \dots + b^8 + 1. \end{aligned}$$

Clearly  $\alpha\beta \neq \alpha$ . Thus  $\alpha$  is not a universally invariant element of BG.

Take  $\alpha = \{1 + a + b + \dots + ab^8 + ab + ab^2 + \dots + ab^8\}$  in BG. We see  $\alpha\beta = \alpha$  for every  $\beta \in BG \setminus \{0\}$ . This  $\alpha$  is a universally invariant element of BG.  $0 \in BG$  is also universally invariant element of BG as  $0.\alpha = 0$  for every  $\alpha \in BG$ .

It is pertinent to mention here that when B is any other Boolean algebra of higher order say ( $|B| \geq 4$ ) then we cannot say BG will contain universally invariant elements other than zero. However we may have invariant elements.

**Example 7.21:** Let  $G = \langle g \mid g^4 = 1 \rangle$  be the cyclic group of order four.  $B = \{0, a, b, 1\}$  be the Boolean algebra of order four. Let BG be the GB-algebraic structure. Let  $\alpha = 1 + g + g^2 + g^3$ .  $\alpha$  is an invariant element of BG but  $\alpha$  is not an universally invariant element of BG for it we take  $\beta = ag + bg^2$ .

We see  $\alpha\beta \neq \alpha$ . But  
 $\alpha.(g+g^2) = \alpha$

$\alpha(1+g) = \alpha$  and  
 $\alpha(1+g+g^2) = \alpha$  and so on.

The following problems are left open for an innovative reader to solve.

**Problem 1:** If  $\alpha \in B$  is an universally invariant element of  $BG$  (where  $B = \{0, 1\}$ ) and  $G$  is a finite group does it imply  $\text{cont } \alpha$  is order of  $G$ . ( $G$  a finite group).

**Problem 2:** Is it possible for  $BG$  to have a universally invariant elements if  $|B| \geq 4$ ?

**THEOREM 7.9:** *Let  $G$  be an abelian group such that  $H$  is a proper finite subgroup of  $G$ .*

*$B = \{0, 1\}$  be a Boolean algebra of order 2. Let  $BG$  be the  $GB$ -algebraic structure. Then  $\alpha \in BG$  such that  $\text{cont } \alpha = H$  is a invariant element of  $BG$ .*

**Proof:** Let  $H$  be the finite subgroup of  $G$ , say  $H = \{1, h_1, \dots, h_t\} \subseteq G$ . Take  $\alpha = (1 + h_1 + \dots + h_t) \in BG$  is an invariant element of  $G$  as  $\alpha \cdot \beta = \alpha$  for every  $\beta \in BG \setminus \{0\}$  with  $\text{cont } \beta \subseteq H$ .

Hence the claim.

Now we are interested in studying whether the  $GB$ -algebraic structure  $BG$  contains any other universally invariant elements other than  $\alpha = 1 + g_1 + \dots + g_n$  where  $G = \{1, g_1, \dots, g_n\}$  and  $B = \{0, 1\}$ . ( $0 \in BG$  is trivially universally invariant element of  $BG$  as  $0 \cdot \alpha = 0$  for all  $\alpha \in BG$ ).

**Example 7.22:** Let  $G = D_{2,7} = \{a, b \mid a^2 = b^7 = 1; bab = a\}$  be the dihedral group of order 14.  $B = \{0, 1\}$  be the Boolean algebra of order two  $BG$  the  $GB$  algebraic structure  $BG$  has atleast one universally invariant element and atleast eight invariant elements.

Let  $x = \sum_{ix} g_i \in BG$  with  $\text{cont } \alpha = G$ . Then  $\alpha$  is a universally invariant element of  $BG$ . Let  $S_1 = 1 + a$ ,  $S_2 = 1 + ab$ ,  $S_3 = 1 + ab^2$ ,  $S_4 = 1 + ab^3$ ,  $S_5 = 1 + ab^4$ ,  $S_6 = 1 + ab^5$ ,  $S_7 = 1 + ab^6$

and  $S_8 = 1 + b + b^2 + b^3 + b^4 + b^5 + b^6$  we see  $(1+a)a = 1+a = S_1$  so is invariant.

$(1+ab)(ab) = 1 + ab$  so  $S_2$  is invariant. Likewise we can show these eight element in BG are invariant elements of BG. It is left for the reader to find other invariant elements of BG.

**DEFINITION 7.8:** Let  $G$  be a non commutative group and  $B$  any Boolean algebra. Let  $BG$  be the GB-algebraic structure. We define center of BG denoted by  $C(BG) = \{x \in BG / x\alpha = \alpha x \text{ for all } \alpha \in BG\}$ .

**Example 7.23:** Let  $G = S_3$  be the symmetric group of degree three and  $B = \{0, 1\}$  be the Boolean algebra of order two.

$$C(BG) = \{0, 1, 1+p_1 + p_2 + p_3 + p_4 + p_5, 1 + p_4 + p_5, p_1 + p_2 + p_3, 1 + p_1 + p_2 + p_3, p_1 + p_2 + p_3 + p_4 + p_5, p_4 + p_5\}$$

We see  $o(BG) = 2^6$  where as  $o(C(BG)) = 2^3$ . Further  $C(BG)$  is a GB-subalgebraic structure of BG.

Let  $G$  be a commutative group  $B$  any Boolean algebra  $BG$  be the GB-algebraic structure. Clearly  $C(BG) = BG$  we know if  $G$  is a non commutative group then  $C(BG) \subseteq BG$ .

Now we find out whether  $C(BG)$  has any nice algebraic structure. To be more precise. Can  $C(BG)$  be a GB-algebraic substructure of BG?

In view of this we define the following new notion.

**DEFINITION 7.9:** Let  $G$  be any group and  $B$  a Boolean algebra.  $BG$  the GB-algebraic structure. Let  $P$  be a proper subset of the BG. We say  $P$  is a GB-algebraic pseudo substructure if the following condition are true.

- (1)  $0, 1 \in P$ ,
- (2) for every  $\alpha, \beta \in P$ ;  $\alpha + \beta \in P$ ,
- (3) for every  $\alpha, \beta \in P$ ;  $\alpha\beta$  and  $\beta\alpha \in P$ .



**THEOREM 7.10:** Let  $G$  be a group.  $B$  a Boolean algebra.  $BG$  be the  $GB$ -algebraic structure.  $C(BG)$  be the center of  $BG$ .  $C(BG)$  is a  $GB$ -algebraic pseudo substructure of  $BG$ .

**Proof:** Let  $C(BG) = \{\alpha \in BG / x \alpha = \alpha x \text{ for all } x \in BG\}$  be the center of  $BG$ . To prove  $C(BG)$  is a  $GB$ -algebraic pseudo substructure. Clearly  $0, 1 \in C(BG)$ .

Take  $\alpha, \beta \in C(BG)$ ; to show  $\alpha + \beta \in C(BG)$ . Given  $\alpha x = x \alpha$  and  $\beta x = x \beta$  for all  $x \in BG$ .

$$\begin{aligned} (\alpha + \beta)x &= \alpha x + \beta x = x \alpha + x \beta \\ &= x(\alpha + \beta). \text{ So } \alpha + \beta \in C(BG) \end{aligned}$$

Let  $\alpha, \beta \in C(BG)$  to show  $\alpha\beta \in C(BG)$ .

Now to show  $(\alpha\beta)x = x(\alpha\beta)$  for all  $x \in BG$ .

$$\begin{aligned} (\alpha\beta)x &= \alpha(\beta x) = \alpha(x\beta) \\ &= (\alpha x)\beta = (x\alpha)\beta = x(\alpha\beta) \end{aligned}$$

Thus  $\alpha\beta \in C(BG)$ .

Thus  $C(BG)$  is a  $GB$ -algebraic pseudo substructure of  $BG$ .

**Example 7.24:** Let  $B = \{0, 1\}$  be the Boolean algebra of order two.  $A_4$  be the alternating subgroup of  $S_4$ .  $BA_4$  be the  $GB$ -algebraic structure. To find  $C(BA_4)$ . Clearly  $0, 1 \in C(BA_4)$ .

$$\begin{aligned} \text{Take } \alpha &= 1 + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} + \\ &\quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \in BA_4. \end{aligned}$$

We see  $\alpha \in C(BA_4)$ .

Let  $\beta = \sum_{g_i} g_i \in BA_4$  such that  $\text{cont } \beta = A_4$ , then  $\beta \in (BA_4)$ .

Thus  $C(BA_4) = \{0, 1, \alpha, \beta\}$  we see  $C(BA_4)$  is a GB-algebraic pseudo substructure of  $BA_4$ .

Thus we have proved  $C(BG)$  is only a GB-algebraic pseudo substructure of  $BG$ .

It is pertinent to mention here that we have no relation between the GB-algebraic substructure and GB-algebraic pseudo substructure. For we see in case of GB-algebraic pseudo substructure  $P$ , we do not have a subgroup of  $G$  to be a subset of  $P$ . We have of course the two element Boolean algebra to be always a subset of  $P$ . Of course we may have a GB-algebraic substructure to be a proper subset of a GB-algebraic pseudo substructure and vice versa. To this end we give some examples.

**Example 7.25:** Let  $G = D_{2.8} = \{a, b \mid a^2 = b^8 = 1; bab = a\}$  be the dihedral group.  $B = \{0, 1\}$  be the Boolean algebra.  $BG$  is the GB-algebraic structure. Now  $C(BD_{2.8}) = \{0, 1, b^4, \alpha = \sum g_i, (\text{cont } \alpha = D_{2.8}), \beta = 1 + b^4\}$  is a GB-algebraic pseudo substructure of  $BG$ .

Clearly  $C(BD_{2.8})$  is a GB-algebraic substructure for as it contains GB-algebraic substructure namely  $T = \{0, 1, b^4, 1 + b^4\} \subseteq C(BD_{2.8})$ ,  $B = \{0, 1\}$  and  $H = \{1, b^4\}$ .

So study of conditions for a GB-algebraic pseudo substructure to be also a GB-algebraic substructure is an interesting problem.

**Example 7.26:** Let  $G = S_3 \times D_{2.7}$  be a group and  $B = \{0, 1\}$  be a Boolean algebra. We see  $H = \{(1, p_1) \times D_{2.7} \subseteq G$  be the subgroup of  $G$ .  $BH$  is a GB-algebraic substructure of  $BG$  but  $BH$  is not a GB-algebraic pseudo substructure of  $BG$ .

**Example 7.27:** Let  $G = S_3 \times A_4 \times D_{2,7}$  be the group and  $B = \{0, 1\}$  be the Boolean algebra of order two. Let  $BG$  be the GB-algebraic structure.

Let  $H = S_3 \times \{e\} \times \{1\} \subseteq G$ ; then  $BH$  is the GB-algebraic substructure of  $BG$ . For  $H \subseteq G$  is a subgroup and  $B = \{0, 1\}$  is the Boolean algebra. If we take  $K = S_3 \times \{e\} \times D_{2,7}$  the subgroup of  $G$  then  $BK \subseteq BG$  is the GB algebraic substructure of  $BG$ .

We can take  $T = \{e\} \times \{e\} \times D_{2,7} \subseteq G$  then  $BT$  is the GB-algebraic substructure of  $BG$ .

Now having seen substructures of  $BG$  we now proceed onto define the notion of the  $BG$  normalized subalgebraic structure.

**DEFINITION 7.10:** Let  $G$  be any group and  $B$  any Boolean algebra.  $BG$  the GB-algebraic structure. Suppose  $T \subseteq BG$  be a proper GB-algebraic substructure of  $BG$  such that

$gTg^{-1} \subseteq T$  for every  $g \in G$  then we call  $T$  to be a normalized GB-algebraic substructure of  $BG$  or  $T$  is called the normal GB-algebraic substructure of  $BG$ .

We illustrate this by some examples.

**Example 7.28:** Let  $G = A_4$  be the alternating subgroup of  $S_4$ .  $B = \{0, 1\}$  be the Boolean algebra.  $BG$  be the GB-algebraic structure.

$$\text{Take } H = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \right\} \subseteq A_4.$$

$B \subseteq BH$  is a GB-algebraic normal substructure of  $BG$ . Clearly  $g(BH)g^{-1} \subseteq BH$  for every  $g \in A_4$ .

We prove the following important and interesting theorems.

**THEOREM 7.11:** *Let  $G$  be a group having normal subgroups.  $B$  any Boolean algebra,  $BG$  the GB-algebraic structure. If  $H$  is a normal subgroup of  $G$  then  $BH$  is a GB-algebraic normal substructure of  $BG$ .*

**Proof:** Given  $G$  is a group and  $B$  any Boolean algebra.  $BG$  the GB-algebraic structure. Let  $H$  be a nontrivial normal subgroup of  $H$ .  $BH$  is clearly a GB-algebraic substructure of  $BG$ . We see for every  $g \in G$ ,  $gBHg^{-1} \subseteq BH$  as  $gHg^{-1} = H$ . Hence  $BH$  is a GB-algebraic normal substructure of  $BG$ .

**THEOREM 7.12:** *Let  $G$  be a simple group (i.e.,  $G$  has no nontrivial normal subgroups).  $B$  the Boolean algebra of order two then  $BG$  has no GB-normal algebraic substructure.*

**Proof:** If  $G$  is a simple group it clearly implies  $G$  has no normal subgroups. We see if  $T$  is any GB-algebraic substructure then  $T$  should contain a proper subset  $H$  which is a subgroup of  $G$  and or a proper subset  $B_1 \subseteq T$  where  $B_1$  is a Boolean subalgebra of  $G$ . Since  $B$  is a Boolean algebra of order two we cannot find  $B_1 \subseteq B$ . So the only condition for  $T$  to be a GB-algebraic substructure is we need  $T$  to contain a proper subset  $H$  which is a subgroup of  $G$ .

Thus if  $T$  is to be a GB-algebraic normal substructure we need  $gTg^{-1} \subseteq T$  this would be possible only if the subgroup  $H$  of  $G$  is normal in  $G$ .

**Example 7.29:** Let  $G = S_3 \times A_4$  be a group  $B = \{0, 1\}$  be a Boolean algebra.  $BG$  be the GB-algebraic structure.

$$\text{Let } H = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} \right\} \subseteq G$$

be a subgroup of  $G$ . We see  $BH$  is a GB-algebraic substructure of  $BG$  but  $BH$  is not a GB-algebraic normal substructure of  $BG$ .

However in view of this we give the following results.

**THEOREM 7.13:** *Let  $G$  be a simple group.  $B$  a Boolean algebra of order two.  $BG$  the GB-algebraic structure.  $BG$  has GB-algebraic substructure which is not a GB-algebraic normal substructure.*

**Proof:** Without loss of generality we can assume  $BG$  is the GB-algebraic structure in which the group  $G$  has subgroups which are not normal in  $G$ . Let  $H$  be a subgroup of  $G$  which is not a normal subgroup of  $G$ . We see  $BH$  is a GB-algebraic substructure which is clearly not a GB-algebraic normal substructure.

**THEOREM 7.14:** *Let  $G = \langle g \mid g^p = 1 \rangle$  where  $G$  is a cyclic group of order  $p$ ,  $p$  a prime  $B = \{0, 1\}$  be a Boolean algebra of order two we see  $BG$ , the GB-algebraic structure has no proper GB-algebraic substructures and no proper GB-algebraic normal substructures.*

**Proof:** Given  $G$  is a cyclic group of order  $p$ ,  $p$  a prime and  $B = \{0, 1\}$  be the Boolean algebra of order two.  $BG$  is a GB-algebraic structure. If  $H$  is to be any proper GB-algebraic substructure of  $BG$  we must have in  $H$  a proper subset  $P$  in  $H$  such that  $P$  is a proper subgroup of  $G$ . But  $G$  has no proper subgroups, hence  $H$  cannot be a GB-algebraic substructure of  $BG$ .

Hence the claim.

**THEOREM 7.15:** *Let  $G$  be a simple group.  $B$  be a Boolean algebra of order greater than or equal to four. Then the GB-algebraic structure has GB-algebraic normal substructures.*

**Proof:** Given  $B$  is a Boolean algebra of order greater than or equal to four. So  $B$  has  $B_1 = \{0, 1\}$  to be sub Boolean algebra of  $B$ . Take  $B_1G$ ,  $B_1G$  is a GB-algebraic substructure of  $BG$  and  $B_1G$  is such that  $gB_1Gg^{-1} = B_1G$  for every  $g \in G$ .

Hence the claim.

Now having seen some of the properties of GB-algebraic substructure we illustrate them by some examples.

**Example 7.30:** Let  $G = \langle g \mid g^p = 1 \rangle$  be a cyclic group of prime order  $p$  and  $B_1 = \{0, 1, a, b\}$  be a Boolean algebra of order four.  $B_1G$  is a GB-algebraic structure. Take  $B = \{0, 1\} \subseteq B_1$  clearly  $BG$  is a GB-algebraic normal substructure of  $BG$ .

Hence the claim.

**Example 7.31:** Let  $G = D_{2p} = \langle a, b \mid a^2 = b^p = 1, bab = a \rangle$  be the dihedral group of order  $2p$  where  $p$  is a prime.  $B = \{0, 1\}$  be the Boolean algebra of order two.  $BG$  the GB-algebraic structure has GB-algebraic normal substructures.

$H = \{1, b, b_2, \dots, b_{p-1}\}$  be normal subgroup of  $G = D_{2p}$  so  $BH$  is a GB algebraic normal substructure of  $BG$ .

**Example 7.32:** Let  $S_n$  be a symmetric group of degree  $n$ ,  $B = \{0, 1\}$  be the Boolean algebra of order two.  $BS_n$  be the GB-algebraic structure.  $BS_n$  has GB-algebraic normal substructure. Take in  $BA_n$ ,  $A_n$  the normal subgroup of  $S_n$ .  $BA_n$  is a GB-algebraic substructure of  $BS_n$ . Hence  $BA_n$  is a GB algebraic normal substructures of  $BS_n$ .

**THEOREM 7.16:** *Let  $A_n$  be a alternating group ( $n \geq 5$  or  $n = 3$ ). Let  $B = \{0, 1\}$  be a Boolean algebra of order two.  $BA_n$  has no GB-algebraic normal substructures.*

**Proof:** The result follows from the fact that  $A_n$  is simple  $n \geq 5$  or  $n = 3$ . Hence  $BA_n$  has no GB-algebraic normal substructures.

**Corollary:** Let  $A_4$  be the alternating group.  $B = \{0, 1\}$  be the Boolean algebra of order four. Then  $BA_4$  has GB-algebraic normal substructures.

**Proof:** Since  $A_4$  has  $H =$

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \right\}$$

to be normal subgroup of  $A_4$  we have  $BH$  to be GB-algebraic normal substructures of  $BA_4$ .

$$\text{Further take } K = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} \right\} \subseteq A_4;$$

$BK$  is only GB-algebraic substructure of  $BA_4$  and is not a GB-algebraic normal substructure of  $BA_4$ .

## FURTHER READING

1. Baker K.A., *Equation Classes of Modular Lattices*, Pacific. J. Math. 28, 9-15, 1969.
2. Birkhoff G, *Combinatorial Relations in Projective Geometries*, Ann. of Math. 36, 743-748, 1935.
3. Birkhoff G, *Subdirect Unions in Universal Algebra*, Bull. Amer. Math. Soc. 50, 764-768, 1944.
4. Birkhoff G, *Lattices Theory*, 2<sup>nd</sup> Edition, Amer. Math. Soc. Providence R.I. , 1948.
5. Dilworth, R.P., and McLaughlin, J.E., *Distributivity in Lattices*, Duke Math.J.19, 683-693, 1952.
6. Gratzner, G., *Equational Classes of Lattices*, Duke J. Math. 33 613-622, 1966.
7. Gratzner, G., *General Lattice Theory*, Birkhauser Verlag Basel and Stuttgart, 1978.



8. Iqbalunnisa and W.B. Vasantha, *On a Certain Class of Modular Lattices*, J. of Madras Univ, Vol. 41, 166-9, 1978.
9. Iqbalunnisa and W.B. Vasantha, *Super Modular Lattices*, J. of Madras Univ, Vol. 44, 58-80, 1981.
10. Iqbalunnisa, *Lattice Translates and congruences*, J. Indian Math. Soc., Vol. 26, 81-86, 1962.
11. Iqbalunnisa, *On Neutral elements in a lattice*, J. Indian Math. Soc. (NS), Vol. 28, 25-31, 1964.
12. Iqbalunnisa, *On types of lattices*, Fund. Math. Vol. LIX, 97-102, 1966.
13. Johnson, B., *Equational classes of lattices*, Duke. J. Math. 33, 613-622, 1966.
14. Menger, K., *New Foundations of Projective and Affine Geometry*, Annals of Math. 37, 456-482, 1936.
15. Schmidt, E.T., and Gratzer, G., *Ideals and congruence relations in Lattices*, Acta Math. Acad. Sci. Hung. (9), 137-175, 1956.
16. Schmidt, E.T., and Gratzer, G., *Standard Ideals in Lattices*, Acta Math. Acad. Sci. Hung. 12, 17-56, 1961.
17. Wille, R., *Primitive subsets of Lattices*, Algebra Universalis 2, 95-98, 1972.

# INDEX

## A

Additive translate, 10

## C

Center of GB-algebraic structure, 119-121

Congruence relation, 11-12

Content of  $\alpha$ , 109-111

## D

Distributive prime intervals, 15

## E

Equivalence relation, 9

## G

GB-algebraic commutative structure, 102-3

GB-algebraic non commutative structure, 102-3

GB-algebraic pseudo substructure, 119-121

GB-algebraic structure, 99-101

## **L**

Lattice translate of an interval, 10

## **M**

Modular element of a lattice, 81-3

Multiplicative translate, 10

## **N**

Normal GB-algebraic substructure, 122-3

n-semisuper modular lattices, 73-5

## **P**

Perspective intervals, 10

Prime interval, 10

## **S**

Semi-super modular lattices, 51-3

Separable, 14

Simple GB-algebraic structure, 108-9

Simple modular lattice, 25-7

Smarandache lattice, 85-7

Smarandache simple lattice, 95-7

Smarandache strong lattice, 96-7

Smarandache sub lattice, 92-3

Sub interval, 10-1

Super modular lattice, 31-3

## **Z**

Zero divisor in GB-structure, 103-5

## ABOUT THE AUTHORS

**Dr. Iqbal Unnisa** (1936-2011) was born in Meenambur, Gingee Taluk, Tamil Nadu on 21<sup>st</sup> November 1936 to Mr. Mohammed Jalaluddin Kokan and Mrs. Aaisha Begum. She was the first Muslim woman to receive her doctorate degree in the Sciences in India. She obtained her M.Sc. and Doctor of Philosophy (PhD, 1964) from the University of Madras under the guidance of Dr. V. S. Krishnan.

She was the first woman to be Director of the Ramanujan Institute for Advanced Study in Mathematics, University of Madras. After working for more than 30 years in research and teaching in India, she moved to Saudi Arabia, where she served as Professor of Mathematics at King Saud University College of Science, Riyadh. She has published several significant research papers in international journals that are referred in books on Lattice Theory. She has guided six Ph.D. and 20 M.Phil scholars. An excellent researcher and trendsetting administrator, she has left behind a rich legacy that we proudly remember.

---

**Dr.W.B.Vasantha Kandasamy** is an Associate Professor in the Department of Mathematics, Indian Institute of Technology Madras, Chennai. In the past decade she has guided 13 Ph.D. scholars in the different fields of non-associative algebras, algebraic coding theory, transportation theory, fuzzy groups, and applications of fuzzy theory of the problems faced in chemical industries and cement industries. She has to her credit 646 research papers. She has guided over 68 M.Sc. and M.Tech. projects. She has worked in collaboration projects with the Indian Space Research Organization and with the Tamil Nadu State AIDS Control Society. She is presently working on a research project funded by the Board of Research in Nuclear Sciences, Government of India. This is her 75<sup>th</sup> book.

On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

She can be contacted at [vasanthakandasamy@gmail.com](mailto:vasanthakandasamy@gmail.com)

Web Site: [http://mat.iitm.ac.in/home/wbv/public\\_html/](http://mat.iitm.ac.in/home/wbv/public_html/)

or <http://www.vasantha.in>

---

**Dr. Florentin Smarandache** is a Professor of Mathematics at the University of New Mexico in USA. He published over 75 books and 200 articles and notes in mathematics, physics, philosophy, psychology, rebus, literature. In mathematics his research is in number theory, non-Euclidean geometry, synthetic geometry, algebraic structures, statistics, neutrosophic logic and set (generalizations of fuzzy logic and set respectively), neutrosophic probability (generalization of classical and imprecise probability). Also, small contributions to nuclear and particle physics, information fusion, neutrosophy (a generalization of dialectics), law of sensations and stimuli, etc. He got the 2010 Telesio-Galilei Academy of Science Gold Medal, Adjunct Professor (equivalent to Doctor Honoris Causa) of Beijing Jiaotong University in 2011, and 2011 Romanian Academy Award for Technical Science (the highest in the country). Dr. W. B. Vasantha Kandasamy and Dr. Florentin Smarandache got the 2011 New Mexico Book Award for Algebraic Structures. He can be contacted at [smarand@unm.edu](mailto:smarand@unm.edu)

---

**In this book the authors introduce a new class of lattices called super modular lattices which is an equational class of lattices lying between the equational class of distributive lattices and modular lattices. Several other new properties related with these lattices are introduced, described and developed.**

US \$35.00  
Educational  
Publisher Inc.

ISBN 9781599731957



9 781599 731957